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Geometry of Generic Isolated Horizons

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Abstract

Geometrical structures intrinsic to non-expanding, weakly isolated and isolated horizons are analyzed and compared with structures which arise in other contexts within general relativity, e.g., at null infinity. In particular, we address in detail the issue of singling out the preferred normals to these horizons required in various applications. This work provides powerful tools to extract invariant, physical information from numerical simulations of the near horizon, strong field geometry. While it complements the previous analysis of laws governing the mechanics of weakly isolated horizons, prior knowledge of those results is not assumed.

I. INTRODUCTION

Isolated horizons approximate event horizons of black holes at late stages of gravitational collapse and of black hole mergers when back-scattered radiation falling into the hole can be neglected [1]. However, unlike event horizons, they are defined quasi-locally and, unlike Killing horizons, they do not require the presence of a Killing vector in their neighborhood. Therefore isolated horizons can be easily located, e.g., in numerical simulations [2]. They can be rotating and may be distorted due to the presence of other black holes, matter discs, external magnetic fields, etc. Consequently, the isolated horizon framework can serve as a

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powerful tool in a variety of physical situations. (For early work in which similar ideas were explored from somewhat different perspectives, see [3–5].)

The notion of an isolated horizon was first introduced to generalize the laws governing black hole mechanics to more realistic situations which allow for gravitational (and other) radiation in the exterior region of space-time [6–8]. It has since proved to be useful also in several other contexts, ranging from numerical relativity to background independent quantum gravity: i) it plays a key role in an ongoing program for extracting physics from numerical simulations of black hole mergers [1,2,9]; ii) it has led to the introduction [10,7,11] of a physical model of hairy black holes, systematizing a large body of results obtained from a mixture of analytical and numerical investigations; and, iii) it serves as a point of departure for statistical mechanical entropy calculations in which all non-rotating black holes (extremal or not) *and* cosmological horizons are incorporated in a single stroke [13,12,14].

The purpose of this paper is to analyze in detail the intrinsically defined, geometrical structures on non-expanding, weakly isolated and isolated horizons and to study their interplay with Einstein's equations (possibly with matter sources). The key geometric structures consist of null normals ℓ^a, the intrinsic (degenerate) metric q_{ab} and the derivative operator $\mathcal{D}$, induced by the space-time connection ∇. We will analyze relations among them which follow directly from the definitions of these horizons; derive the constraints they must satisfy as a consequence of the pull-back of the field equations to horizons; specify the "free data"; and spell out the information about *space-time* curvature contained in their intrinsic geometry. In addition, we will address an issue that plays an important role in applications of the framework, in particular to numerical relativity [2]: Can we use the intrinsic structures to select a preferred class of null normals ℓ to the horizon? We will show that, generically, the answer is in the affirmative. Overall, the results of this paper complement those on mechanics of isolated horizons [6–8]. Whereas in that work the emphasis was on the infinite dimensional phase space of space-times admitting isolated horizons as null boundaries, in the present case we focus on the geometry of isolated horizons in individual space-times.

In Section II we consider non-expanding horizons (NEHs) $\triangle$. These are 3-dimensional, null sub-manifolds of space-time with vanishing expansion on which a mild energy condition holds. We show they are naturally equipped with a (degenerate) metric q and a derivative operator $\mathcal{D}$ and analyze the restrictions the space-time curvature must satisfy on them. In Section III we define weakly isolated horizons (WIHs) $(\triangle, [\ell])$ by imposing a restriction on an equivalence class $[\ell]$ of future directed null normals ℓ to $\triangle$ (where two are equivalent if they are *constant* multiples of each other). The pull-back of the field equations to $\triangle$ constrain the geometrical pair $(q, \mathcal{D})$. We analyze these constraint equations and spell out the freely specifiable parts of these variables. Finally, we show that in the case when the surface gravity is non-zero, the WIHs $(\triangle, [\ell])$ admit a natural foliation which can, in particular, be used in numerical relativity to construct invariantly defined coordinates and tetrads. A WIH has certain similarities with null infinity since both can be regarded as null boundaries. We show that the intrinsic geometric structure of WIHs and its interplay with field equations is rather similar to that at null infinity in absence of radiation.

A natural question is whether every NEH $\triangle$ admits a null normal ℓ such that $(\triangle, [\ell])$ is a WIH and if the choice is unique. The answer to the existence question is in the affirmative. However, the choice is far from being unique. In Section IV, therefore, we consider additional geometrical conditions on ℓ which *can* select $[\ell]$ uniquely. It turns out there is an obvious

choice which fulfills this task. Although this choice seems natural at first, in the case when Δ *is* a Killing horizon, the restriction of the Killing field to the horizon need not belong to the equivalence class $[\ell]$ so chosen. Furthermore, in this case, $[\ell]$ need not even be preserved by the isometry generated by the Killing field. We then assume that the pull-back of the space-time Ricci tensor to $\triangle$ satisfies a natural condition and introduce a more sophisticated restriction on $[\ell]$. Not only does it select $[\ell]$ uniquely in generic situations but it is also free from the above drawbacks. Thus, on a generic WIH, there is a way to select a canonical $[\ell]$ *using only the intrinsically available structure* on $\triangle$ such that, in the case of a Killing horizon, $[\ell]$ consists precisely of the constant multiples of the Killing vector.

On any WIH, the flow generated by $[\ell]$ preserves the metric and also parts of the connection $\mathcal{D}$. These conditions are sufficient for generalizing black hole mechanics [6–8]. However, from the geometric perspective of this paper, it is more natural to impose a stronger requirement and demand that the flow of $[\ell]$ preserve the full connection $\mathcal{D}$. This condition defines the isolated horizons (IHs) of Section V. The additional conditions are true restrictions in the sense that, while every NEH can be made a WIH by selecting an appropriate null normal, the same is *not* true of IHs. Again, we analyze the constraints imposed by the field equations, isolate the 'free data' and point out that, if a NEH geometry admits two IH *non-extremal* IH structures, they are related by a (geometry preserving) diffeomorphism. Thus, if it exists, a non-extremal IH structure is unique. In Section VI we consider the analytic extension of $\triangle$ and its intrinsic geometry and present a more geometric characterization of the canonical normals $[\ell]$. Section VII summarizes our main conclusions and suggests potential applications of our results.

In Appendix A we obtain some explicit conditions that a NEH geometry must satisfy if it is to admit an IH structure and if it is to admit two distinct IH structures, one extremal and the other non-extremal. Generically the IH structure is unique; the exceptional cases are now even more restricted than they were in Section IV B. Up to this point, we work with tensors in index-free or Penrose's abstract index notation [15]. For the convenience of readers more familiar with the Newman-Penrose framework [16,15], in Appendix B we derive the main results as well as a few other interesting facts in that framework.

II. NON-EXPANDING HORIZONS

In this section, we introduce the notion of non-expanding horizons (NEH) and review some of their properties. This discussion sets the stage for the main definitions introduced later in the paper. Most of the properties summarized here were discussed in [6]. They are included here for completeness and will be presented from a more geometric perspective. For an extension of a part of the framework to general, null hypersurfaces, see [5,17].

A. Preliminaries

Consider a 4-dimensional space-time (M, g) and a 3-dimensional, null sub-manifold $\triangle$ thereof. We will denote a(n arbitrary) *future-directed* null normal to $\triangle$ by ℓ.

Definition 1: $\triangle$ will be called a *non-expanding horizon* if
i) $\triangle$ is diffeomorphic to the product $\hat{\triangle} \times \mathbb{R}$ where $\hat{\triangle}$ is a 2-sphere, and the fibers of the

projection

$$\Pi \ : \ \hat{\triangle} \times \mathrm{I\!R} \to \hat{\triangle}$$

are null curves in $\triangle$;
ii) the expansion of any null normal ℓ to $\triangle$ vanishes; and,
iii) Einstein's equations hold on $\triangle$ and the stress-energy tensor T_{ab} is such that $-T^a{}_b\ell^b$ is causal and future-directed on $\triangle$.
Note that if these conditions hold for one choice of null normal, they hold for all. Condition *iii*) is very mild; in particular, it is implied by the (much stronger) dominant energy condition satisfied by the Klein-Gordon, Maxwell, dilaton, Yang-Mills and Higgs fields as well as by perfect fluids.

Condition *i*) above implies $\triangle$ is ruled by the integral curves of the null direction field which is normal to it. For later purposes, it is useful to introduce an equivalence relation:

$$\ell' \ \sim \ \ell \text{ whenever } \ \ell' \ = \ c\ell, \quad \text{ where } \ c = \text{const.}$$

and denote the equivalence classes by $[\ell]$. Using the common terminology at null infinity, the integral curves of $[\ell]$ will be called *generators* of $\triangle$ and $\hat{\triangle}$ will be called the *base space* of $\triangle$. *i*) also implies that the generators of $\triangle$ are geodesic. Given ℓ, a function

$$v \ : \ \triangle \ \to \ \mathrm{I\!R}, \ \text{ such that } \quad \mathcal{L}_\ell \, v = 1$$

will be called a *compatible coordinate*. Finally, the Raychaudhuri equation and condition *iii*) imply that ℓ is also *shear-free* on $\triangle$. Hence, condition *ii*) may be replaced by:
ii)$'$ The null direction tangent to $\triangle$ is covariantly constant on $\triangle$.
Similarly, *i*) and *ii*) may be replaced by:
$i, ii)''$ $\triangle$ is isometric to the orthogonal product of a 2-sphere $\hat{\triangle}$ equipped with a positive definite metric $\hat{q}_{ab}$ and the line $\mathrm{I\!R}$ equipped with the trivial, zero 'metric'.
In Definition 1, we chose the formulation in terms of notions that are most commonly used in the relativity literature.

Let us now examine the geometry of non-expanding horizons (NEHs). First, via pull-back, the space-time metric g_{ab} induces a 'metric' q_{ab} on $\triangle$ with signature (0,+,+). Since q is degenerate, there exist infinitely many (torsion-free) derivative operators on $\triangle$ which annihilate it. However, the Raychaudhuri equation and fact that $\triangle$ is divergence-free and matter satisfies our energy condition imply that $\triangle$ is also shear-free. We will now show that the vanishing of expansion and shear in turn imply that we can uniquely select a preferred derivative operator intrinsic to $\triangle$. Let us first note that the condition (ii') implies the space-time parallel transport restricted to the curves tangent to $\triangle$ preserves the tangent bundle $T(\triangle)$ to $\triangle$. Indeed, at each point $x \in \triangle$, the tangent space $T_x\triangle$ is the subspace of T_xM orthogonal to ℓ, and[1]

$$(X^a \nabla_a Y^b)\ell_b \, \hat{=} \, -X^a Y^b \nabla_a \ell_b \, \hat{=} \, 0$$

[1]Throughout this paper $\hat{=}$ stands for "equals, on $\triangle$, to" and, unless otherwise stated, d will denote the exterior derivative intrinsic to $\triangle$.

for every $X, Y \in T(\triangle)$, where ∇ is the derivative operator on M compatible with g. Therefore, ∇ induces a derivative operator $\mathcal{D}$ on $\triangle$ via

$$X^a D_a Y^b = X^a \nabla_a Y^b \tag{2.1}$$

for all vector fields X, Y tangential to $\triangle$. The operator $\mathcal{D}$ can be extended, in the standard fashion, to the covectors K_a defined intrinsically on $\triangle$: $(\mathcal{D}_a K_b) Y^b = \mathcal{D}_a (K_b Y^b) - K_b (\mathcal{D}_a Y^b)$. Finally, the action of $\mathcal{D}$ can be uniquely extended by the Leibnitz rule to arbitrary tensor fields defined intrinsically on $\triangle$. Pulling back the equation $\nabla_a g_{mn} \hat{=} 0$ to $\triangle$ we obtain $\mathcal{D}_a q_{mn} \hat{=} 0$; $\mathcal{D}$ is compatible with the degenerate metric on $\triangle$.

By *geometry of $\triangle$ we will mean the pair $(q, \mathcal{D})$ consisting of the intrinsic metric q and the derivative $\mathcal{D}$ on $\triangle$*. Because ℓ is shear- and expansion-free, $\mathcal{L}_\ell\, q \hat{=} 0$ for any null normal ℓ to $\triangle$. Furthermore, by construction, $q_{ab} \ell^b \hat{=} 0$. Hence we conclude that q_{ab} is the pull-back to $\triangle$ of a Riemannian metric $\hat{q}_{ab}$ on the 2-sphere $\hat{\triangle}$ of generators of $\triangle$:

$$q \hat{=} \Pi^\star\ \hat{q}$$

Similarly, the natural (area) 2-form $\hat{\epsilon}$ compatible with $\hat{q}$ on $\hat{\triangle}$ can be pulled-back to yield a natural 2-form

$$ {}^2\!\epsilon \hat{=} \Pi^\star\ \hat{\epsilon}$$

on $\triangle$, which will turn out to be useful. Since $\hat{q}$ is non-degenerate, it defines an unique (torsion-free) derivative operator $\hat{\mathcal{D}}$ on $\hat{\triangle}$. If h is the pull-back to $\triangle$ of any covector $\hat{h}$ on $\hat{\triangle}$, we have

$$\mathcal{D}_a\, h_b = \Pi^\star\, (\hat{\mathcal{D}}_a\, \hat{h}_b) \tag{2.2}$$

Therefore, in particular, we have:

$$\mathcal{D}_a\, {}^2\!\epsilon_{bc} \hat{=} 0\,.$$

However, $\mathcal{D}$ is *not* determined by $\hat{\mathcal{D}}$; it has 'more information'. In particular, $\hat{\mathcal{D}}$ does not constrain the action of $\mathcal{D}$ on ℓ which determines 'surface gravity' and 'gravitational angular momentum' of $\triangle$ [8]. Let us extract this part of the extra information. Since $X^a Y^b \nabla_a \ell_b \hat{=} 0$ for all X, Y tangential to $\triangle$, there exists a 1-form ω on $\triangle$ such that

$$\mathcal{D}_a \ell^b \hat{=} \omega_a \ell^b\,. \tag{2.3}$$

By construction, ω is tied to a choice of ℓ. (Strictly, we should denote it as $\omega_{(\ell)}$ but will refrain from doing so for notational simplicity.) Under the rescaling $\ell \mapsto f\ell$, we have

$$\omega \mapsto \omega + d \ln f\,. \tag{2.4}$$

The function $\kappa_{(\ell)}$ defined by

$$\kappa_{(\ell)} := \omega_a \ell^a$$

will be called *surface gravity* of $\triangle$ relative to the null normal ℓ. Under the rescaling $\ell \mapsto f\ell$, we have

$$\kappa_{(\ell)} \mapsto \kappa_{(\ell)} + \mathcal{L}_\ell \ln f. \tag{2.5}$$

Hence, given only a NEH, it is not meaningful to ask if the surface gravity is constant on Δ, i.e., if the zeroth law of black hole mechanics holds. It can hold for one choice of ℓ but not for another. This brings out the fact that the notion of a non-expanding horizon does not capture even the most basic *physical* structure available on the event horizon of a black hole in equilibrium. In Section III we will strengthen the definition by adding suitable conditions to ensure that the zeroth law does hold.

B. Space-time curvature on $\triangle$

The geometry $(q, \mathcal{D})$ of $\triangle$ determines the pull-back $R^a{}_{\underleftarrow{bcd}}$ (on the three covariant indices) of the space-time Riemann tensor $R^a{}_{bcd}$, and, by algebraic symmetries of $R^a{}_{bcd}$, the pull-back $R_{\underleftarrow{ab}}$ of the Ricci tensor R_{ab}. Our energy condition (in Definition 1) and the Raychaudhuri equation on ℓ not only force the shear of ℓ to vanish, but also imply the Ricci tensor must satisfy:

$$R_{ab}\ell^a\ell^b \hat{=} 0. \tag{2.6}$$

Our energy condition then further implies that $R_{ab}\ell^b$ is proportional to ℓ_a, that is

$$R_{ab}\ell^a X^b \ \hat{=}\ 0, \tag{2.7}$$

for any vector field X tangent to $\triangle$. In the Newman-Penrose notation (see Appendix B), this result can be stated as follows: in a null frame $(m,\, \bar{m},\, n,\, \ell)$,

$$\Phi_{00} \ \hat{=}\ \Phi_{01} \ = \bar{\Phi}_{10} \hat{=} 0. \tag{2.8}$$

(This equation, in turn, constrains the matter fields via $T_{ab}\ell^a X^b \hat{=} 0$. However, in what follows we are primarily interested in the geometrical fields.) The remaining components of $R_{\underleftarrow{ab}}$ enter 'constraint equations' on $\triangle$ and are discussed in Section III B.

Let us next consider the Weyl tensor. Because ℓ is expansion and shear-free, it must lie along one of the principal null directions of the Weyl tensor (see Appendix B or [6]). Then, equation (2.7) implies ℓ in fact lies along a *double* principal null direction of the Weyl tensor [6]. In the Newman-Penrose notation we have:

$$\Psi_0 \ \hat{=}\ \Psi_1 \hat{=} 0, \tag{2.9}$$

whence Ψ_2 is gauge invariant (i.e., independent of the choice of the null-tetrad vectors $(n, m, \bar{m})$) on $\triangle$. The remaining components of $R^a{}_{\underleftarrow{bcd}}$ will not be needed in the main text but are given at the end of the subsection 2.a of the Appendix B for completeness.

Next, let us explore the relation between the intrinsic derivative operator $\mathcal{D}$ on $\triangle$ and the (non-vanishing components of the) Weyl curvature. Note first that while the 1-form ω of (2.3) depends on the choice of ℓ, it is clear from its transformation property (2.4) that its exterior derivative $d\omega$ is in fact independent of ℓ. Furthermore, a simple calculation [7] shows $d\omega$ can be expressed in terms of the Riemann curvature. Using (2.3) and (2.7) we have

$$d\omega \;\hat{=}\; 2\mathrm{Im}(\Psi_2)\,{}^2\epsilon. \tag{2.10}$$

As one's experience with the Newman Penrose framework would suggest, $\mathrm{Im}(\Psi_2)$ captures the gravitational contribution to the angular-momentum at $\triangle$ [8]. Therefore, we will refer to $\mathrm{Im}(\Psi_2)$ as the *rotational curvature scalar* and ω as the *rotational 1-form potential.*

Using the Cartan identity and (2.10), the Lie derivative of ω with respect to ℓ is given by

$$\mathcal{L}_\ell \omega_a \;\hat{=}\; 2\mathrm{Im}(\Psi_2)\,\ell^b\,{}^2\epsilon_{ba} + \mathcal{D}_a(\ell^b\omega_b) \;\;\hat{=}\;\; \mathcal{D}_a\kappa_{(\ell)}. \tag{2.11}$$

Recall that $\kappa_{(\ell)}$ is associated with a pair $(\triangle, \ell)$ rather than with the 3-manifold $\triangle$ itself and it changes under rescalings of ℓ via (2.5). Eq (2.11) provides a necessary and sufficient condition on the choice of ℓ to ensure that $\kappa_{(\ell)}$ is constant on $\triangle$, i.e., the zeroth law holds. It will motivate our definition of weakly isolated horizons in the next section.

Finally, we will show that the rotational curvature scalar $\mathrm{Im}(\Psi_2)$ also admits a *scalar* potential, which turns out to be useful. Note first that (2.10) and (2.11) imply $\mathrm{Im}(\Psi_2)$ has an unambiguous projection to $\hat{\triangle}$:

$$\mathcal{L}_\ell\; \mathrm{Im}(\Psi_2) \;\;\hat{=}\, 0,$$

and, considered as a function defined on $\hat{\triangle}$, it satisfies a 'global constraint', namely

$$\int_{\hat{\triangle}} 2\mathrm{Im}(\Psi_2)\,\hat{\epsilon} \;=\; 0. \tag{2.12}$$

Therefore, there exists a well-defined *rotational scalar potential* U such that

$$\hat{\Delta} U \;=\; 2\mathrm{Im}(\Psi_2), \tag{2.13}$$

on $\hat{\triangle}$ which is unique up to an additive constant: the only freedom in the choice of U is

$$U \mapsto U' \;=\; U + U_0, \;\;\; U_0 \;=\; \text{const.}$$

III. WEAKLY ISOLATED HORIZONS $[\ell]$

As noted above, on a NEH there is a large freedom in the choice of null normals corresponding to changes $\ell \mapsto \ell' \;\hat{=}\; f\ell$ with f arbitrary positive function on $\triangle$. Applications of this framework, e.g. to black hole mechanics and numerical relativity, often require the horizon to be equipped with a preferred equivalence class $[\ell]$ [1,6–8]. Therefore, in this section, we will endow NEHs with a specific $[\ell]$ satisfying a weak restriction, which enables one to extend zeroth and first laws of black hole mechanics, and study the resulting geometrical structures.

This section is divided into three parts. In the first, we introduce the basic definition of a weakly isolated horizon (WIH); in the second, we examine the interplay between the space-time geometry and the intrinsic structures on WIHs; and, in the third, we show that (non-extremal) WIHs admit a preferred foliation. These structures are useful not only for mathematical physics but also for numerical relativity [1].

A. Preliminaries

Definition. A *weakly isolated horizon* $(\triangle, [\ell])$ is a non-expanding horizon $\triangle$, equipped with an equivalence class $[\ell]$ of null normals (under *constant* rescaling) such that the flow of ℓ preserves the rotation 1-form potential ω (of (2.3))

$$\mathcal{L}_\ell \, \omega_a \ \hat{=} \ 0. \tag{3.1}$$

If this condition holds for one ℓ, it holds for all ℓ in $[\ell]$.

Note that, by definition, a WIH is equipped with a specific equivalence class $[\ell]$ of null normals. Recall that, on a space-like hypersurface S, the extrinsic curvature can be defined as $K_a{}^b = \nabla_{\underleftarrow{a}} n^b$ where n is the unit normal and under-arrow denotes the pull-back to S. A natural analog of the extrinsic curvature on a WIH is then $L_a{}^b \hat{=} \mathcal{D}_a \ell^b$ and, by virtue of (2.3), (3.1) is equivalent to requiring that $L_a{}^b$ be Lie-dragged along ℓ.[2] Thus, while on a NEH only the intrinsic metric q is "time-independent", on a WIH the analog of extrinsic curvature is also required to be "time-independent". In this sense, while NEHs resemble Killing horizons only to first order, WIH resemble a Killing horizon also to the next order. Note however that the full connection $\mathcal{D}$ or curvature components such as Ψ_4 *can be* time-dependent on a WIH.

Eqs (3.1) and (2.11) imply $\kappa_{(\ell)}$ is constant on $\triangle$. Thus *the zeroth law of black hole mechanics naturally extends to* WIHs. However, since a WIH is equipped only with an equivalence class $[\ell]$ of null normals, where $\ell \sim \ell'$ if and only if $\ell' = c\ell$ with c a positive constant, and since $\kappa_{(l')} = c\kappa_{(\ell)}$ by (2.5), that surface gravity does not have a canonical value on WIHs unless it vanishes. Thus, WIHs naturally fall in to two classes: i) non-extremal, in which case the surface gravity for every ℓ in $[\ell]$ is non-zero; and, ii) extremal, in which case it is zero.

Given any NEH $\triangle$, we can always choose a null normal $[\ell]$ such that $(\triangle, [\ell])$ is an extremal WIH. In this case, $\ell^a \nabla_a \ell^b \hat{=} 0$; integral curves of $\ell \in [\ell]$ are *affinely parametrized geodesics*. Fix an ℓ_0 which is affinely parameterized on $\triangle$ and denote by v_0 a compatible coordinate (so $\mathcal{L}_{\ell_0} v_0 \hat{=} 1$). On the same manifold $\triangle$, consider any other null normal ℓ'_0 such that $(\triangle, [\ell'_0])$ is also an extremal WIH. Let v'_0 be a compatible coordinate for ℓ'_0. Then, it is straightforward to check that

$$\ell'^a \hat{=} (\frac{1}{A})\ell^a \quad \text{and} \quad v'_0 \hat{=} A v_0 + B \tag{3.2}$$

for some functions A and B on $\triangle$ such that $\mathcal{L}_\ell A \hat{=} \mathcal{L}_\ell B \hat{=} 0$ and $A > 0$. Thus, every NEH admits a family of null normals $\ell, \ell', ...$ such that $(\triangle, [\ell])$, $(\triangle, [\ell'])$, ... are all extremal WIHs and any two of these null normals are related by (3.2).

Next, let us examine the rescaling of ℓ_0 which maps the fiducial, extremal WIH to any given, non-extremal WIH $(\triangle, [\ell])$ with surface gravity $\kappa_{(\ell)}$ (see (2.5)). It is given by:

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[2] Of course, on a null surface there is no canonical analog of extrinsic curvature. For example, since the pull-back of ℓ_a to $\triangle$ vanishes, $L_{ab} := \nabla_{\underleftarrow{a}} \ell_b$ vanishes identically. Thus, the index structure of $L_a{}^b$ has to be chosen carefully if one wishes to strengthen the notion of a NEH.

$$\ell^a \mathrel{\hat{=}} \kappa_{(\ell)}\,(v_0 - B)\,\ell_0^a \quad \text{and} \quad \kappa_{(\ell)} v = \ln\, \kappa_{(\ell)}(v_0 - B) \tag{3.3}$$

where $\mathcal{L}_\ell B \mathrel{\hat{=}} 0$. Thus, every non-extremal WIH $(\triangle, [\ell])$ is obtained from the fiducial ℓ_0 via (3.3). To summarize, simply by restricting the null normals ℓ to lie in a suitably chosen equivalence class $[\ell]$, from any given NEH $\triangle$ we can construct a WIH $(\triangle, [\ell])$ which is either extremal or non-extremal. However, because of the arbitrary functions involved in (3.2) and (3.3), *there is an infinite dimensional freedom in this construction.*

We will conclude this sub-section with four remarks.

i) We could also begin with a non-extremal WIH and construct an extremal one. Let ℓ be a null normal to $\triangle$ such that $\kappa_{(\ell)} = \text{const} \not\mathrel{\hat{=}} 0$. Fix any function v on $\triangle$ with $\mathcal{L}_\ell v \mathrel{\hat{=}} 1$. Then, if we set $\ell' \mathrel{\hat{=}} (\exp\, -\kappa_{(\ell)} v) B\, \ell$ where B is any function on $\triangle$ satisfying $\mathcal{L}_\ell B \mathrel{\hat{=}} 0$, then $(\triangle, [\ell'])$ is an extremal WIH and every extremal WIH can be obtained via this construction.

ii) Note that in (3.3), ℓ vanishes if $v_0 = B$. Since in the main text, for simplicity, we have restricted ourselves only to future directed, non-zero null normals, strictly speaking, the construction breaks down and only a portion of $(\triangle, [\ell_0])$ can be regarded as a non-extremal WIH. However, geometrically, there is no a priori obstruction to allow ℓ to vanish somewhere and consider the entire solution (3.3). Then ℓ changes orientation at points $v_0 = B$. In such situations, the section of $\triangle$ defined by $\ell = 0$ will be called the *crossover section of* ℓ (which may not have a 2-sphere topology if ℓ_0 fails to be complete on $\triangle$). Notice however that from the perspective of the geometry of the NEH $\triangle$, there is nothing special about the crossover section of a given WIH. Indeed, *every* section of $\triangle$ is a crossover section of some non-extremal WIH. In Section VI, we will consider the geodesically complete, analytic extension $\bar{\triangle}$ of $\triangle$. In such an extension, a non-extremal WIH always contains a *2-sphere* cross-section on which ℓ^a vanishes. It will be called a *cross-over sphere.* In the Kruskal extension of the Schwarzschild space-time, the Killing horizon bifurcates at the cross-over sphere.

iii) Suppose a WIH $(\triangle, [\ell])$ is complete in the sense that each $\ell \in [\ell]$ is a (future directed, nowhere vanishing) complete vector field. Then, given a representative $\ell \in [\ell]$, the corresponding rotation 1-form ω may be thought of as an Abelian connection on the trivial bundle $\triangle \to \hat{\triangle}$, where the structure group is (the additive group of reals) given by the flow generated by ℓ.

iv) Let $\triangle_K$ be a Killing horizon for Killing fields $[\xi]$, where, as before, the square brackets denote equivalence class of vector fields where any two are related through rescaling by a positive constant. If we set $[\ell] = [\xi]$, then $(\triangle, [\ell])$ is a WIH. Thus, the passage from the NEH to a WIH can be understood as follows: whereas on a NEH we only ask that the null normal be a Killing field to the first order (i.e., it Lie drag the intrinsic metric q_{ab} on $\triangle$), on a WIH, the permissible null normals mimic the Killing fields in a stronger sense; they Lie drag also the connection 1-form ω.

B. Constraint equations and free data on WIHs

On a space-like 3-manifold, the 4-geometry induces an intrinsic metric and an extrinsic curvature and these are subject to the well-known constraint equations. Under the weak assumption that space-time admits constant mean curvature slices, one can find the freely

specifiable data through the Lichnerowicz-York construction [19]. Similarly, at null infinity, $\mathcal{I}$, the (conformally rescaled) 4-metric naturally induces a triplet $(q, n, \mathcal{D})$ consisting of a degenerate metric q, null normal n^a and an intrinsic derivative operator $\mathcal{D}$ [20]. These fields capture the information contained in the pull-back of the 4-Ricci tensor to $\mathcal{I}$, and five of the ten components, Ψ_4, Ψ_3 and $\mathrm{Im}\,\Psi_2$, of the Weyl tensor. The constraint equations they satisfy enable one to isolate the radiative degrees of freedom of the gravitational field. We will now carry out a similar analysis at weakly isolated horizons. More precisely, we ask: what are the constraints which the triplet $(q_{ab}, [\ell^a], \mathcal{D})$ must satisfy and can they be solved to obtain the "freely specifiable data" at WIHs?

The definition of a WIH immediately leads to the first set of equations:

$$\begin{aligned} q_{ab} l^b &\,\widehat{=}\, 0, \qquad \mathcal{L}_\ell\, q_{ab} \,\widehat{=}\, 0 \\ \mathcal{D}_a \ell^b &\,\widehat{=}\, \omega_a \ell^b, \quad \mathcal{L}_\ell \omega_a \,\widehat{=}\, 0. \end{aligned} \tag{3.4}$$

Since q is degenerate, it does not fully determine $\mathcal{D}$. Nonetheless, as noted in (2.2), it does constrain $\mathcal{D}$: if h is the pull-back to $\triangle$ of a 1-form field $\hat{h}$ on $\hat{\triangle}$, then

$$\mathcal{D}_a h_b = \Pi^*(\hat{D}_a \hat{h}_b)$$

where $\hat{\mathcal{D}}$ is the unique (torsion-free) derivative operator on $\hat{\triangle}$ compatible with $\hat{q}$. Therefore, given q, to specify the action of $\mathcal{D}$ on an arbitrary co-vector field W on $\triangle$ —and hence on any tensor field on $\triangle$— it is necessary and sufficient to specify its action on a co-vector field n with $n_a \ell^a \,\widehat{\neq}\, 0$. Thus, we only need to specify

$$S_{ab} := \mathcal{D}_a n_b \,.$$

Without loss of generality, we can choose n to satisfy

$$n_a \ell^a \,\widehat{=}\, -1; \qquad \text{and} \qquad \mathcal{D}_{[a} n_{b]} \,\widehat{=}\, 0\,. \tag{3.5}$$

(Note that (3.5) is equivalent to setting $n = dv$, with v a compatible coordinate for ℓ.) These properties (3.5) now imply that S_{ab} is symmetric and satisfies

$$S_{ab} \ell^b \,\widehat{=}\, \omega_a \tag{3.6}$$

Hence, given (q, ω), to specify $\mathcal{D}$, it suffices to provide just the projection $\tilde{S}_{ab}$ of S_{ab} on 2-sphere cross-sections of $\triangle$ orthogonal to n:

$$\tilde{S}_{ab} \,\widehat{=}\, \tilde{q}_a{}^c \tilde{q}_b{}^d S_{cd}$$

where $\tilde{q}_a{}^c$ is the projection operator on these 2-spheres (satisfying $\tilde{q}_a{}^b \ell^a \,\widehat{=}\, 0 \,\widehat{=}\, \tilde{q}_a{}^b n_b$, and $\tilde{q}_a{}^b \tilde{X}^a = \tilde{X}^b$ for all $\tilde{X}^b$ tangential to the 2-sphere cross-sections.) The trace $\tilde{q}^{ab} \tilde{S}_{ab} =: 2\mu$ of $\tilde{S}_{ab}$ represents the "transversal" expansion of the 2-spheres while its trace-free part λ_{ab} represents their "transversal" shear (where "transversal" refers to n). Thus, we have shown that the geometry $(q, \mathcal{D})$ of $(\triangle, [\ell])$ is completely specified by $(q, \omega, \tilde{S}_{ab})$. The question now is: What are the restrictions imposed on this triplet by the field equations $R_{ab} = 8\pi G(T_{ab} - \frac{1}{2} T g_{ab})$ at $\triangle$?

It turns out that the components of R_{ab} transversal to $\triangle$ dictate the "evolution" of fields off $\triangle$ while the pull-back of R_{ab} constrains its intrinsic geometry $(q, \mathcal{D})$. A direct calculation yields: $\underleftarrow{R_{ab}}\ell^b \,\widehat{=}\, 2\ell^a \mathcal{D}_{[a}\omega_{b]}$. But we have already seen that $\underleftarrow{R_{ab}}\ell^b$ vanishes identically on every NEH (see (2.7). Using (3.1) we conclude that this vanishing implies and is implied by the zeroth law, i.e., the condition that $\kappa_{(\ell)} = \omega_a \ell^a$ is constant on $\triangle$. (The field equations also constrain matter fields in the obvious way but these restrictions are not relevant for $(q, \omega, \tilde{S})$.) Next, another direct calculation yields

$$\mathcal{L}_\ell \, \tilde{S}_{ab} \,\widehat{=}\, -\kappa_{(\ell)} \tilde{S}_{ab} + \tilde{\mathcal{D}}_{(a} \tilde{\omega}_{b)} + \tilde{\omega}_a \tilde{\omega}_b - \frac{1}{2} \tilde{\mathcal{R}}_{ab} + \frac{1}{2} \tilde{q}_a{}^c \, \tilde{q}_b{}^d \, R_{cd} \tag{3.7}$$

where $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{R}}_{ab}$ denote the derivative operator and the Ricci tensor on the 2-sphere cross-sections defined by n_a and $\tilde{\omega}_a$ is the projection of ω_a on these cross-sections. Thus, by (2.7) the contraction of the pulled-back Ricci tensor with ℓ vanishes, while by (3.7) its remaining components serve as the source of the time derivative of $\tilde{S}_{ab}$. Hence, the constraint equations on $(q, \mathcal{D}) \equiv (q, \omega, \tilde{S})$ are simply the zeroth law the

$$\kappa_{(\ell)} \,\widehat{=}\, \mathrm{const} \tag{3.8}$$

and the equation

$$\mathcal{L}_\ell \, \tilde{S}_{ab} = -\kappa_{(\ell)} \tilde{S}_{ab} + \tilde{\mathcal{D}}_{(a} \tilde{\omega}_{b)} + \tilde{\omega}_a \tilde{\omega}_b - \frac{1}{2} \tilde{\mathcal{R}}_{ab} + 4\pi G \tilde{q}_a{}^c \, \tilde{q}_b{}^d \, (T_{cd} - \frac{1}{2} T q_{cd}) \,, \tag{3.9}$$

where we have used the field equations in the last term. Having these constraints at our disposal, we can now spell out the freely specifiable data.

Suppose we are given a pair, $(\triangle, [\ell])$, (satisfying condition *i*) of Definition 1) and the pull-back and the trace of the stress-energy tensor T_{ab} on $\triangle$. To construct the permissible pairs $(q_{ab}, \mathcal{D})$ such that $(\triangle, [\ell])$ is a weakly isolated horizon, we proceed as follows. Choose any 2-sphere cross section $\tilde{\triangle}$ of $\triangle$ and fix a Riemannian metric $\tilde{q}_{ab}$, a 1-form $\tilde{\omega}_a$, and a symmetric tensor $\tilde{S}_{ab}$ on $\tilde{\triangle}$. Extend these fields to all of $\triangle$ in two stages: First, set $\ell^a \tilde{q}_{ab} = 0 = \ell^a \tilde{S}_{ab}$ and $\ell^a \omega_a = \kappa_{(\ell)}$ at $\tilde{\triangle}$, with $\kappa_{(\ell)}$ an arbitrary positive constant. Second, carry these fields to other points $\triangle$ by setting $\mathcal{L}_\ell \tilde{q}_{ab} = 0 = \mathcal{L}_\ell \tilde{\omega}_a$ and using (3.9). Then, we have a permissible pair $(q = \tilde{q}, \mathcal{D})$, where $\mathcal{D}$ is constructed from $\tilde{q}$, $\tilde{\omega}$ and $\tilde{S}$. All permissible pairs can be constructed in this way. Thus, given $(\triangle, [\ell])$ the free data consist of a constant $\kappa_{(\ell)}$ and fields $\tilde{q}_{ab}, \tilde{\omega}_a, \tilde{S}_{ab}$ on any 2-sphere cross-section of $\triangle$. In the Newman-Penrose notation, $\tilde{\omega}_a$ is specified by the spin coefficient π and $\tilde{S}_{ab}$ by the coefficients μ and λ.

We conclude this sub-section with four remarks.

i) In the standard 3+1 decomposition of space-time by spatial sub-manifolds, the pull-back of the space-time Ricci tensor to the sub-manifolds yields the evolution equations along the normal to the surface. In the present case, since the normal ℓ is also tangential to $\triangle$, the analogous equation (3.7) is now a constraint.

ii) At null infinity, $\mathcal{I}$, because of the available conformal freedom, we can choose the analog of q to be the standard 2-sphere metric, and the analog of ω as well as the trace of the analog of $\tilde{S}_{ab}$ to be zero. The (conformal equivalence class) of the derivative operator $\mathcal{D}$ at $\mathcal{I}$ is thus fully determined by the transversal shear. In absence of radiation, we again have an analog of (3.9), but it now says that the transversal shear is *time-independent*. Thus,

while the overall mathematical structure is parallel, the detailed conclusions are different because of differences in the boundary conditions in the two regimes.

iii) Since the above construction involved the choice of a cross-section $\tilde{\triangle}$ of $\triangle$, there is a certain gauge freedom in the free data. Suppose we determine the geometry of $\triangle$ by choosing a $\tilde{\triangle}$ and carrying out the construction to obtain a triplet (q, ω, S). Then, if we choose a new cross-section $\tilde{\triangle}'$ and use as initial data the fields induced on $\tilde{\triangle}'$ by (q, ω, S), our construction will yield new fields (q', ω', S') on $\triangle$, gauge related to the original triplet. The gauge transformations are:

$$q'_{ab} = q_{ab}, \quad \omega'_a = \omega_a, \quad n'_a = n_a + \mathcal{D}_a f, \quad S'_{ab} = S_{ab} + \mathcal{D}_a \mathcal{D}_b f$$

for some function f satisfying $\mathcal{L}_\ell f \,\hat{=}\, 0$. Hence, $\tilde{\omega}$ and $\tilde{S}_{ab}$ transform via

$$\tilde{\omega}'_a = \tilde{\omega}_a - \kappa_{(\ell)} \mathcal{D}_a f \quad \text{and} \quad \tilde{S}'_{ab} = \tilde{S}_{ab} + \mathcal{D}_a \mathcal{D}_b f \,. \tag{3.10}$$

In spite of this non-trivial transformation property, $\mathcal{L}_\ell \tilde{S}_{ab} \,\hat{=}\, \mathcal{L}_\ell \tilde{S}'_{ab}$. This fact will play an important role in the next two sections. Finally, we will show in Section III C that on non-extremal WIHs this gauge freedom can be completely eliminated by a natural choice of cross-sections.

iv) Let us suppose the pull-back of the space-time Ricci tensor to $\triangle$ is time-independent, i.e., $\mathcal{L}_\ell R_{\underleftarrow{ab}} \,\hat{=}\, 0$. (Incidentally, because of (2.7), if this property holds for one null normal to $\triangle$, it holds for all.) Then all but the first term on the right side of (3.7) are "time independent" whence, we can easily integrate this equation. If $\kappa_{(\ell)}$ is non-zero, the solution is:

$$\tilde{S}_{ab} = e^{-\kappa_{(\ell)} v}\, \tilde{S}^0_{ab} + \frac{1}{\kappa_{(\ell)}} \left[\tilde{\mathcal{D}}_{(a} \tilde{\omega}_{b)} + \tilde{\omega}_a \tilde{\omega}_b - \frac{1}{2} \tilde{\mathcal{R}}_{ab} + \frac{1}{2} \tilde{q}_a{}^c\, \tilde{q}_b{}^d\, R_{cd} \right] \tag{3.11}$$

for some v-independent $\tilde{S}^0_{ab}$, while if $\kappa_{(\ell)}$ vanishes, it is:

$$\tilde{S}_{ab} = \tilde{S}^0_{ab} + \left[\tilde{\mathcal{D}}_{(a} \tilde{\omega}_{b)} + \tilde{\omega}_a \tilde{\omega}_b - \frac{1}{2} \tilde{\mathcal{R}}_{ab} + \frac{1}{2} \tilde{q}_a{}^c\, \tilde{q}_b{}^d\, R_{cd} \right] v \tag{3.12}$$

Thus, in either case, $\tilde{S}_{ab}$ has a very specific time dependence[3]. Let us suppose that the WIH $(\triangle, \ell)$ is complete and non-extremal. Then $\tilde{S}_{ab}$ diverges at one end unless $\tilde{S}^0_{ab}$ vanishes identically. By remark iii) above, this property is independent of the choice of n. However, it does depend on the choice of ℓ. In Section IV B we will use this fact to select a canonical $[\ell]$ on a generic NEH $\triangle$.

C. Good cuts of a non-extremal WIH $(\triangle, [\ell])$

We will now show that non-extremal WIHs admit a natural foliation which can be regarded as providing a 'rest frame' for the horizon. Fix a WIH with $\kappa_{(\ell)} \not\hat{=}\, 0$. Since by the

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[3]The possibility of time dependence of these fields brings out the generality of the notion of WIHs. On a Killing horizon, by contrast, every geometrical field is time independent.

definition of a WIH $\mathcal{L}_\ell d\omega \,\hat{=}\, 0$, and since (2.10) implies that the contraction of ℓ with $d\omega$ vanishes, $d\omega$ has an unambiguous projection to the 2-sphere $\hat{\triangle}$ of generators of $\triangle$. Since this projection is necessarily an exact 2-form, there exists a 1-form $\hat{\omega}$ defined globally on $\hat{\triangle}$ such that

$$\Pi^\star \hat{d}\hat{\omega} \;=\; d\omega.$$

While $\hat{\omega}$ is *not* unique, *any* choice will give $d(\omega - \Pi^\star\hat{\omega}) \,\hat{=}\, 0$ and $\ell^a(\omega_a - \Pi^\star\hat{\omega}_a) = \ell^a\omega_a = \kappa_{(\ell)}$. Thus, there exists a function v, defined globally on $\triangle$, such that

$$\omega \;=\; \Pi^\star\hat{\omega} + \kappa_{(\ell)} dv. \tag{3.13}$$

Clearly, v is determined up to an additive constant and is a coordinate compatible with ℓ (i.e., $\mathcal{L}_\ell v = 1$). Hence, $v =$ const surfaces define a foliation of $\triangle$. (In the extremal case, $\omega = \Pi^\star\hat{\omega}$ and we can no longer extract the function v or the foliation.) Thus, each choice of $\hat{\omega}$ provides a foliation and the issue is if there is a natural, invariant choice.

The answer is in the affirmative. Using the 2-metric $\hat{q}_{ab}$ and the Hodge star $\hat{\star}$ on $\hat{\triangle}$, any 1-form $\hat{\omega}$ can be uniquely decomposed in to exact and a co-exact parts:

$$\hat{\omega} \;=\; -\hat{\star}\hat{d}U \;+\; \hat{d}p,$$

where U, p are smooth, real functions on $\hat{\triangle}$. U is the rotational scalar potential defined in Section II B, while p represents the gauge-freedom in the choice of $\hat{\omega}$. It is natural to set it to zero, so that

$$\hat{\omega} \;=\; -\hat{\star}\hat{d}U. \tag{3.14}$$

This prescription determines $\hat{\omega}$ uniquely. Furthermore, this is a natural choice because, e.g., when the rotational curvature scalar $\mathrm{Im}(\Psi_2)$ vanishes, so does $\hat{\omega}$ in this gauge. The corresponding $v =$ const sections will be called *good cuts*. This foliation *always* exists on non-extremal WIHs and is invariantly defined in the sense that it can be constructed entirely from structures naturally available on $(\triangle, [\ell])$. In particular, if the space-time admits an isometry which preserves the given WIH, good cuts are necessarily mapped in to each other by that isometry.

We conclude with two remarks.

i) In the definition of a WIH, we have not required the vector fields $[\ell]$ to be complete. This is because, in the physically interesting situations, one expects the isolated horizon to be formed at some finite time by a dynamical process and hence incomplete at least in the past. When $[\ell]$ is incomplete, the good cuts defined above need not be global cross-sections of $\triangle$. However the requirement that $\hat{\omega}$ is of the form (3.14) globally on $\hat{\triangle}$ still distinguishes the good cut foliation and the associated coordinates v uniquely.

ii) At null infinity, in absence of radiative modes, it is also possible to single out "good cuts" [20]. On WIHs, by definition, there is no gravitational radiation. Is there then a close relation between the two constructions? At first sight, the answer may seem to be in the negative because, whereas the emphasis at $\mathcal{I}$ is generally on finding cross sections on which the transversal shear vanishes [18], we made no reference to the shear of n. (Indeed, as the transformation property (3.10) shows, the trace-free part of $\tilde{S}_{ab}$ can not be made to vanish

on a general WIH.) Therefore, the two constructions may appear to be unrelated. However, this is not the case. In both cases, the absence of gravitational radiative modes on the null 3-surface imposes a restriction on $\mathrm{Im}(\Psi_2)$, enabling one to impose natural restrictions on its potential to select good cuts. At null infinity, the potential happens to be the transversal shear while in our case it happens to be the 1-form ω. Thus, when formulated in terms of curvatures and connections intrinsic to the null surfaces [20], the two constructions are parallel. There is nonetheless one residual difference. Whereas the intrinsic degenerate metric on $\mathcal{I}$ has considerable conformal freedom, that on $\triangle$ is unique. Because of this difference in 'rigidity', whereas (in absence of radiative modes) there is a 4-parameter family of good-cuts on $\mathcal{I}$, $\triangle$ is equipped with a 1-parameter family.

IV. PREFERRED CLASSES $[\ell]$ OF NULL NORMALS

In applications of this framework, e.g. to numerical relativity, one can find NEHs but it is generally essential to single out a canonical equivalence class of null-normals $[\ell]$. In the case when the NEH arises as a Killing horizon $\triangle_K$, the Killing vector ξ which is normal to $\triangle_K$ provides a canonical $[\ell] = [\xi]$. (Note that, without recourse to global considerations, ξ is known only up to a multiplicative constant.) This choice turns out to be the appropriate one for many applications. The question is if we can select a canonical $[\ell]$ on a generic NEH such that, in the case of a Killing horizon, the canonical $[\ell]$ agrees with $[\xi]$. In Section IV A, we consider an 'obvious' restriction which suffices to select $[\ell]$ uniquely. While this procedure is rather natural from the standpoint of null geometry, it turns out not to be satisfactory for various applications. In Section IV B, therefore, we strengthen the notion of 'isolation' slightly by requiring that $q^{ab}R_{ab} \mathrel{\hat=} 16\pi G T_{ab}\ell^a n^b$ be time-independent on $\triangle$ and introduce a more sophisticated strategy which is also natural and appropriate for applications.[4]

A. Canonical choice of $[\ell]$ on an extremal WIH

From the perspective of null surfaces, it is natural to limit the freedom in the choice of ℓ by first demanding that it be *affinely parametrized*. Then, $\kappa_{(\ell)}$ vanishes, whence $(\triangle, [\ell])$ is an extremal horizon. However, as we saw in Section III A, there is still an infinite dimensional freedom in the choice of such $[\ell]$. We will now show that this remaining freedom can be eliminated by imposing a natural geometric condition.

Let $\triangle$ be a NEH and let ℓ be a null normal such that $\kappa_{(\ell)} \mathrel{\hat=} 0$. Since $\mathcal{L}_\ell \omega_a$ and $\omega_a \ell^a$ both vanish on $\triangle$, we have

$$\omega \; = \; \Pi^* \hat{\omega},$$

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[4] Here q^{ab} is any 'inverse' of q_{ab}, i.e., any tensor field defined intrinsically on $\triangle$ satisfying $q^{ab}q_{am}q_{bn} \mathrel{\hat=} q_{mn}$. There is a freedom to add to q^{ab} a term $X^{(a}\ell^{b)}$, and to n^a a vector field h^a, where X^a is *any* vector field tangential to $\triangle$ and h^a is any vector field tangential to $\triangle$ satisfying $\ell \cdot h \mathrel{\hat=} -1$. However, our additional requirement is insensitive to this freedom because $R_{ab}l^bX^b$ vanishes on any $\triangle$.

whence the 1-form $\hat{\omega}$ on $\hat{\triangle}$ is defined uniquely by ω on $\triangle$. (In this case, the function p in the decomposition (3.13) no longer represents a gauge freedom; in fact, $\hat{dp}$ is now an invariant of the WIH under consideration.) Now, under a rescaling $\ell \mapsto \ell' \,\hat{=}\, C\ell$ by a function C, with $\mathcal{L}_\ell C \,\hat{=}\, 0$, we have:

$$\omega' \;\hat{=}\; \omega + d\ln C, \qquad p' = p + \ln C. \tag{4.1}$$

Therefore, to select ℓ uniquely up to a multiplicative constant, it is necessary and sufficient to impose a condition which selects p up to an additive constant. Following our strategy of Section III B, we can achieve this by requiring that $\hat{\omega}$ be divergence free on $\hat{\triangle}$. To summarize, *any NEH $\triangle$ admits an unique, extremal WIH structure $(\triangle, [\ell])$ such that:*

$$\kappa_{(\ell)} \;= 0, \quad \hat{d}\,\hat{\star}\hat{\omega} = \hat{\mathrm{div}}\hat{\omega} \;\hat{=}\; 0 \tag{4.2}$$

Purely from geometric considerations intrinsic to $\triangle$, then, we already have a prescription to select the equivalence class $[\ell]$ of null normals uniquely.

Unfortunately, this prescription is not very useful in practice. Suppose $\triangle$ is a Killing horizon for a Killing vector ξ defined in its neighborhood such that $\kappa_{(\xi)}$ is *non-zero*, i.e., the Killing horizon is non-extremal. (This would, in particular, be the case for the Schwarzschild horizon.) Then, not only will the unique equivalence class $[\ell]$ given by the above construction fail to coincide with the natural choice $[\xi]$ but more importantly, because of (3.3), $[\ell]$ would not even be left invariant under the action of the isometry generated by ξ. Indeed, from the 4-dimensional space-time point of view, the assignment of *any* extremal $[\ell]$ to this Killing horizon would be unnatural. We therefore need a more sophisticated strategy to single out a canonical $[\ell]$ on a NEH $\triangle$. In particular, this choice should be left invariant by all isometries preserving $\triangle$.

B. Canonical choice of $[\ell]$ on generic NEHs

Let us now restrict ourselves to NEHs $\triangle$ such that the space-time Ricci tensor satisfies $(\mathcal{L}_\ell R_{ab})q^{ab} \,\hat{=}\, 0$. Via field equations, this condition is equivalent to $\mathcal{L}_\ell(T_{ab}\ell^a n^b) \,\hat{=}\, 0$, where q^{ab} is any "inverse" of q_{ab}. (As in footnote 4, these conditions are independent of the choice of q^{ab} and n^a.) In this sub-section, we will show $\triangle$ generically admits an unique $[\ell]$ such that the transversal expansion μ is "time independent".

Let us choose any null normal ℓ to $\triangle$ and consider the commutator $[\mathcal{L}_\ell, \mathcal{D}]$. Due to general properties of these two operators, there exists a tensor field $C^c_{ab} \,\hat{=}\, C^c_{(ab)}$ on $\triangle$ such that

$$[\mathcal{L}_\ell, \mathcal{D}_a]K_b \,\hat{=}\, C^c_{ab}K_c \tag{4.3}$$

for any covector field K_a on $\triangle$. Property (2.2) implies $C^c_{ab}h_c \,\hat{=}\, 0$ for any h_c defined intrinsically on $\triangle$ satisfying $\ell^a h_a \,\hat{=}\, 0$. Hence, C^c_{ab} has the form:

$$C^c_{ab} \,\hat{=}\, -N_{ab}\ell^c$$

for some symmetric tensor field N_{ab}. Conditions $\mathcal{D}_a\ell^b \,\hat{=}\, \omega_a\ell^b$ and $\mathcal{L}_\ell\omega_b \,\hat{=}\, 0$ now imply

$$\ell^a N_{ab} = 0.$$

Note that under constant rescalings $\ell \mapsto c\ell$, N_{ab} remains unchanged, although under a rescaling by a function, it does change. Therefore, a natural strategy to restrict the choice of ℓ is to impose conditions on N_{ab}. In this sub-section we will show that, on a generic NEH $\triangle$ on which the Ricci tensor satisfies our 'time-independence' condition, one can indeed select $[\ell]$ uniquely by requiring N_{ab} to be trace-free, i.e.,

$$q^{ab} N_{ab} \,\hat{=}\, 0\,. \tag{4.4}$$

(Again the condition is independent of the choice of q^{ab} because $N_{ab}\ell^b \,\hat{=}\, 0$). Thus, a generic NEH $\triangle$ admits a unique WIH structure $(\triangle, [\ell])$ for which N_{ab} is transverse and traceless.

Condition (4.4) has a simple interpretation in terms of structures introduced in Section III B. Choosing $K_a = n_a$ in (4.3), with $\ell^a n_a \,\hat{=}\, -1$ and $\mathcal{L}_\ell n_a \,\hat{=}\, 0$, we find

$$N_{ab} = \mathcal{L}_\ell(\mathcal{D}_a n_b) \equiv \mathcal{L}_\ell S_{ab}.$$

Thus (4.4) requires $q^{ab}\, \mathcal{L}_\ell S_{ab} = 0$. If we require $dn = 0$ as in Section III B, the condition reduces to $\mathcal{L}_\ell \mu \,\hat{=}\, 0$. Thus, our result will establish that *a generic NEH admits a unique* $[\ell]$ *such that* $(\triangle, [\ell])$ *is a WIH on which the transversal expansion* μ *is time independent.* As remarked in Section III B, this condition is independent of the choice of n satisfying (3.5).

Consider any NEH $\triangle$ and introduce on it a fiducial null normal ℓ. Without loss of generality, we can assume ℓ is non-extremal: $\kappa_{(\ell)} \,\hat{\neq}\, 0$. Suppose $N_{ab}q^{ab} \,\hat{\neq}\, 0$. Our task is to find another null normal $\ell' = f\ell$ for which $(\triangle, [\ell'])$ is a WIH with $N'_{ab}q^{ab} \,\hat{=}\, 0$. A simple calculation yields

$$f N'_{ab} \,\hat{=}\, f N_{ab} + 2\omega_{(a}\mathcal{D}_{b)} f \; + \mathcal{D}_a \mathcal{D}_b f. \tag{4.5}$$

Since N_{ab} and N'_{ab} are transversal to ℓ and $\ell' = f\ell$, the functional form of f is severely constrained. Indeed, contracting (4.5) with ℓ^a, one obtains

$$\mathcal{D}_a(\mathcal{L}_\ell f + \kappa_{(\ell)} f) \,\hat{=}\, 0,$$

which can be readily solved to conclude that the f we are seeking has a specific form:

$$f = B e^{-\kappa_{(\ell)} v} + \frac{\kappa_{(\ell')}}{\kappa_{(\ell)}} \qquad \text{where } \mathcal{L}_\ell B \,\hat{=}\, 0 \tag{4.6}$$

Thus our task is to find a B such that $N'_{ab}q^{ab} \,\hat{=}\, 0$.

Let us now introduce a covector field n on $\triangle$ satisfying (3.5), contract (4.5) with the inverse metric $\tilde{q}^{ab}$ satisfying $\tilde{q}^{ab} n_b \,\hat{=}\, 0$, and use the above form of f. The requirement $N'_{ab}q^{ab} \,\hat{=}\, 0$ will be met if and only if

$$[\tilde{\mathcal{D}}^2 + 2\tilde{\omega}^a \mathcal{D}_a + \kappa_{(\ell)}\, \tilde{q}^{ab}\tilde{S}_{ab} + \tilde{q}^{ab} N_{ab}]\, B \;\hat{=}\; -\left(\frac{\kappa_{(\ell')}}{\kappa_{(\ell)}} e^{\kappa_{(\ell)} v}\right) \tilde{q}^{ab} N_{ab} \tag{4.7}$$

We can hope to solve this elliptic equation to determine B. However, there is a potential problem in this strategy: While there is both an explicit time dependence (through $\exp \kappa_{(\ell)} v$) on the right hand side of this equation and an implicit time dependence (via $\tilde{S}_{ab}$) on the

left hand side, the B we are seeking is required to be time-independent. Therefore, it is not immediately clear whether (4.7) admits *any* time-independent solution B.

Let us therefore examine the v dependence in detail. Fortunately, under our assumption that $q^{ab}R_{ab}$ be time independent, we have an explicit formula for $\tilde{q}^{ab}\tilde{S}_{ab}$ (see (3.11))q. Substituting it in (4.7) and using $N_{ab} = \mathcal{L}_\ell S_{ab}$, we obtain

$$\mathbf{M}\, B := \left[\tilde{\mathcal{D}}^2 + 2\tilde{\omega}^a\tilde{\mathcal{D}}_a + \tilde{\mathcal{D}}^a\tilde{\omega}_a + \tilde{\omega}_a\tilde{\omega}^a - \frac{1}{2}\tilde{\mathcal{R}} + \frac{1}{2}\tilde{q}^{ab}\, R_{ab}\right]\, B \;=\; \kappa_{(\ell')}\tilde{q}^{ab}\tilde{S}^0_{ab} \tag{4.8}$$

where $\tilde{S}^0_{ab}$ satisfies $\mathcal{L}_\ell\tilde{S}^0_{ab} \,\hat{=}\, 0$. Thus, (thanks to our assumption $\mathcal{L}_\ell(q^{ab}R_{ab}) \,\hat{=}\, 0$), the elliptic equation on B is in fact v-independent. Hence, we can hope to find a time independent solution B as required. By assumption, the right side is non-zero (for, if it were zero, our fiducial $[\ell]$ already satisfies our condition (4.4)). Therefore, if zero is *not* an eigenvalue of the elliptic operator $\mathbf{M}$, (4.8) is guaranteed to admit a solution which, furthermore, is unique. One can show that the dimension of the kernel of $\mathbf{M}$ is a property only of the NEH $\triangle$ and does not depend on the choice of ℓ or of the cross-section $\tilde{\triangle}$ used to construct $\mathbf{M}$.

We will say that *a NEH is* generic *if the elliptic operator* $\mathbf{M}$ *on* $\hat{\triangle}$ *has trivial kernel* for *some* choice of ℓ with $\kappa_{(\ell)} \not\hat{=}\, 0$ (and $\mathcal{L}_l R_{ab}q^{ab} \,\hat{=}\, 0$ holds). Then, on generic NEHs, there is exactly one $[\ell']$ for which $(\triangle, [\ell'])$ is a WIH with trace-free N'_{ab} (or, time-independent transversal expansion μ). This establishes our assertion. (We will obtain an alternate and geometrically more transparent uniqueness result in section VI.) Furthermore, the generic property guarantees that $\kappa_{(\ell')}$ is non-zero, i.e., the WIH so selected is non-extremal. Finally, note that even if $\mathbf{M}$ has a non-trivial kernel, the right side of (4.8) may be in the image of $\mathbf{M}$. In this case, preferred $[\ell]$s would exist but would not be unique.

Remark: If $\triangle$ is a Killing horizon for a Killing field $[\xi]$, and we choose $[\ell] = [\xi]$, the Killing vector, $\mu = \tilde{q}^{ab}N_{ab}$ is guaranteed to be time-independent. Thus, generically, our condition (4.4) extracts from the Killing property just the "right" information to select a canonical $[\ell]$. The detailed strategy is rather subtle. For example, although it seems natural at first, on generic NEHs we cannot require that all of $\tilde{S}_{ab}$ be time-independent. The rescaling of ℓ provides a single free function B and we can be adjust it to make only the trace time-independent.

Note however, that the existence of a Killing vector in the neighborhood of the horizon is not sufficient to guarantee that $\mathbf{M}$ is invertible. This is in particular the case for extremal Killing horizons $\triangle_K$, for some Killing vector ξ defined near $\triangle$. For, in this case, there obviously does exist a ℓ' for which N'_{ab} is trace-free (namely $\ell' \,\hat{=}\, \xi$) so there is indeed a non-trivial B relating this ℓ' and the fiducial, non-extremal ℓ we began with. Since $\kappa_{(\ell')} \,\hat{=}\, 0$, from (4.8) we conclude that this B is in the kernel of $\mathbf{M}$. Thus, in this case, $\mathbf{M}$ is necessarily non-invertible, whence any extremal isolated horizon is non-generic in the present terminology.

V. ISOLATED HORIZONS.

In this section, we will strengthen the notion of isolation by requiring the intrinsic metric q and the *full* derivative operator $\mathcal{D}$ (rather than just the 1-form ω) be time-independent. We will first introduce the basic definition and then isolate the free data and comment on

the issue of existence and uniqueness of an IH structure on a given NEH. These issues are further discussed in some detail in Appendix A.

Definition: An isolated horizon (IH) is a pair $(\triangle, [\ell])$, where $\triangle$ is a NEH equipped with an equivalence class $[\ell]$ of null normals such that

$$[\mathcal{L}_\ell,\, \mathcal{D}] \,\widehat{=}\, 0\,. \tag{5.1}$$

If this condition holds for one ℓ it holds for all ℓ in $[\ell]$.

Let $\triangle$ be a NEH with geometry $(q, \mathcal{D})$. We will say that this geometry *admits an isolated horizon structure* if there exists a null normal ℓ satisfying (5.1). This IH structure will be said to be extremal if $\kappa_{(\ell)} \,\widehat{=}\, 0$ and non-extremal otherwise. Intuitively, a NEH is an IH if the entire geometry $(q, \mathcal{D})$ of the NEH is 'time-independent'. From the perspective of the intrinsic geometry, this is a stronger and perhaps more natural notion of 'isolation' than that captured in the definition of a WIH. Indeed the basic condition (3.1) in the definition of a WIH can be reformulated as

$$[\mathcal{L}_\ell, \mathcal{D}]\ell^a \,\widehat{=}\, 0,$$

i.e., as restricting the action of the left side of (5.1) to ℓ. In the terminology introduced in Section IV B, an IH is a WIH on which the field N_{ab} vanishes identically; on a WIH only $N_{ab}\ell^b = \mathcal{L}_\ell S_{ab}\ell^b = \mathcal{L}_\ell \omega_a$ vanishes. An IH mimics properties of a Killing horizon to a slightly higher degree than a WIH. However, explicit examples [23] as well as an analysis [21] using the initial value problem based on two null surfaces [22] shows there is an infinite-dimensional class of other examples. In particular, while all geometric fields are time-independent on a Killing horizon, the field Ψ_4, for example, can be time-dependent on a generic IH.

We saw in Section III that any NEH can be made a WIH simply by choosing an appropriate class $[\ell]$ of null normals. The situation with IHs, by contrast, is quite different. Not every NEH admits a null normal satisfying (5.1); this condition is a *genuine* restriction. Indeed, we saw in Section IV B that, generically, weak isolation and the condition $N_{ab}q^{ab} \,\widehat{=}\, 0$ exhaust the rescaling freedom in the choice of $[\ell]$. The resulting $[\ell]$ is then the only candidate for an IH structure on the given NEH. However, in general, the resulting $[\ell]$ will not be such that the trace-free part of N_{ab} also vanishes.

Given a candidate for a black hole space-time, it is generally easy to verify whether one's guess for the horizon is in fact a NEH, which can be readily made a WIH simply by choosing the null normal appropriately. However, it is considerably more difficult to verify whether there exists a null normal which makes it an IH. Necessary conditions for the existence of such a null normal follow from (3.7) and some of these are discussed in Appendix A.

Let us now extract the freely specifiable 'data' on an IH. The structure of a WIH is specified by an equivalence class $[\ell]$ and a pair $(q, \mathcal{D})$, or, as in Section III B, by a constant $\kappa_{(\ell)}$ and a triplet of fields $(q_{ab}, \omega_a, \tilde{S}_{ab})$ satisfying (3.1) and (3.7) on IH. However, since all these fields are time-independent on an isolated horizon, they are constrained further. Let us first consider the case when $\kappa_{(\ell)} \,\widehat{\neq}\, 0$. Then, (3.11) implies $\tilde{S}^0_{ab}$ must vanish, whence $\tilde{S}_{ab}$ is completely determined by q_{ab}, ω_a and the matter fields (ie the pull-back of the four dimensional Ricci tensor to $\triangle$):

$$\tilde{S}_{ab} = \frac{1}{\kappa_{(\ell)}}\left[\tilde{\mathcal{D}}_{(a}\tilde{\omega}_{b)} + \tilde{\omega}_a\tilde{\omega}_b - \frac{1}{2}\tilde{\mathcal{R}}_{ab} + \frac{1}{2}\tilde{q}_a{}^c\,\tilde{q}_b{}^d\,R_{cd}\right] \tag{5.2}$$

Therefore, given $R_{\underleftarrow{ab}}$, to specify the geometry of a non-extremal IH, following the procedure of Section III B, let us fix a 2-sphere cross section $\tilde{\triangle}$ of $\triangle$ and fields (q_{ab}, ω_a) thereon such that: i) the pull-back $\tilde{q}_{ab}$ of q_{ab} is a positive definite metric on $\tilde{\triangle}$, ii) $q_{ab}\ell^a = 0$; and, iii) $\omega_a \ell^a =: \kappa_{(\ell)}$ is a non-zero constant. We then *define* $\tilde{S}_{ab}$ on $\tilde{\triangle}$ via (5.2) and extend all these fields to $\triangle$ by demanding they be Lie-dragged by ℓ. The resulting triplet $(q_{ab}, \omega_a, \tilde{S}_{ab})$ then defines an IH geometry. Thus, relative to the WIH case considered earlier, there is no longer the freedom to choose the transversal expansion μ and shear λ on $\tilde{\triangle}$; these fields are completely determined by the pair (q, ω).

To conclude, let us consider extremal IHs. If $\kappa_{(\ell)} = 0$, $\tilde{S}_{ab}$ is given by (3.12). Isolation implies it is time-independent. Rather than determining $\tilde{S}_{ab}$ in terms of (q_{ab}, ω_a) as in the case $\kappa_{(\ell)} \,\hat{\neq}\, 0$, this condition now implies that (q_{ab}, ω_a) are *themselves constrained* by

$$\tilde{\mathcal{D}}_{(a}\tilde{\omega}_{b)} + \tilde{\omega}_a\tilde{\omega}_b - \frac{1}{2}\tilde{\mathcal{R}}_{ab} + \frac{1}{2}\tilde{q}_a{}^c\,\tilde{q}_b{}^d\,R_{cd} \;=\; 0, \tag{5.3}$$

while $\tilde{S}_{ab}$ is now free (but of course, time independent). Therefore, in this case, the free data consists of triplets $(q_{ab}, \omega_a, \tilde{S}_{ab})$ on $\tilde{\triangle}$, where (q_{ab}, ω_a) are now subject, in addition to the conditions given above (in the $\kappa_{(\ell)} \,\hat{\neq}\, 0$ case), also to (5.3), while $\tilde{S}_{ab}$ is only required to be symmetric and transverse to ℓ. In the Newman-Penrose notation, in this case m_a and π are now subject to (5.3) while μ, λ are now free.

Finally, in the non-extremal case, we can again eliminate the (gauge) freedom in the choice of the cross-section simply by restricting ourselves to the 'good cuts' of Section III C. In this case, $\tilde{\omega}$ is subject to $\tilde{q}^{ab}\tilde{D}_a\tilde{\omega}_b \,\hat{=}\, 0$. Then, given $R_{\underleftarrow{ab}}$, the problem of specification of $(q, \mathcal{D})$ reduces to that of specifying $(\hat{q}, \hat{U})$ on the 2-sphere $\hat{\triangle}$ of generators of $\triangle$, where $\hat{U}$ is the rotational scalar (see Eq(3.14)). The complete diffeomorphism invariant information of an IH structure is encoded in the diffeomorphism class of fields $(\hat{q}, \hat{U})$ on $\hat{\triangle}$. If two distinct IH structures $(\triangle, [\ell], q, \mathcal{D})$ and $(\triangle, [\ell'], q', \mathcal{D}')$ yield the same pair $(\hat{q}, \hat{U})$ on $\hat{\triangle}$, then there is a diffeomorphism from $\triangle$ on to itself which maps $[\ell]$ to $[\ell']$. In this sense, the IH structure is unique in the non-extremal case. The only remaining question is: can a NEH admit two distinct IH structures, one non-extremal and the other extremal? This is possible but, as we show in Appendix A, the corresponding horizon geometry is very severely constrained.

VI. ANALYTIC EXTENSION OF $(\triangle, [\ell], Q, \mathcal{D})$

In the last three sections, we presented constructions to select canonical null normals $[\ell]$ on NEHs. In this section, we will present an alternate characterization of these canonical $[\ell]$ using an analytic extension of $\triangle$ and fields thereon. This characterization is again geometrical and, furthermore, considerably easier to visualize.

Let us begin with some preliminaries. Consider a NEH $\triangle$ and introduce on it an affinely parametrized null normal ℓ_0 and denote the compatible coordinate by v_0. Then $(\triangle, [\ell_0])$ is a an extremal WIH. Therefore, $\mathcal{L}_{\ell_0} q_{ab} \,\hat{=}\, 0$ *and* $\mathcal{L}_{\ell_0}\omega_a \,\hat{=}\, 0$. Furthermore, $\tilde{S}_{ab}$ which captures the remaining information in $\mathcal{D}$ is explicitly given by (3.12). Since, by inspection, $\tilde{S}_{ab}$ is analytic in v_0, and q_{ab} and ω_a are independent of v_0, we can analytically extend $\triangle$ to $\bar{\triangle}$ on which v_0 runs from $-\infty$ to ∞, and extend $(\ell_0, q_{ab}, \mathcal{D})$ to $\bar{\triangle}$. Note that we do *not* assume

that the space-time is analytic even near $\triangle$; we have extended $\triangle$ as an abstract 3-manifold endowed with certain fields.

In this section, we will work with this analytic extension and fields defined intrinsically thereon. The vector field ℓ_0 is complete on $\bar{\triangle}$ and so are *all* other affinely parametrized null normals. Thus, our analytic extension does not depend on the initial choice of the extremal ℓ_0. While these affinely parametrized null normals are nowhere vanishing, non-extremal null normals ℓ can not be everywhere future pointing (see below). Therefore, in this section we will drop the requirement that the null normals be nowhere vanishing and future directed.

We can endow $\triangle$ with *any* non-extremal WIH structure $(\triangle, [\ell])$ with surface gravity $\kappa_{(\ell)}$ by a rescaling

$$\ell_0 \mapsto \ell \mathrel{\hat{=}} \kappa_{(\ell)}(v_0 - B)\,\ell_0 \tag{6.1}$$

for some B satisfying $\mathcal{L}_{\ell_0} B \mathrel{\hat{=}} 0$. Since the surface gravity rescales as $\kappa_{(\ell)} \mapsto c\kappa_{(\ell)}$ under constant rescalings $\ell \mapsto c\ell$ with $c > 0$, the function B is unaffected by these rescalings; thus, there is a 1-1 correspondence between the functions B and non-extremal $[\ell]$. For any given B, it is clear by inspection that $[\ell]$ vanishes on precisely one 2-sphere cross-section $\tilde{\triangle}_\ell$ of $\bar{\triangle}$, given by $v_0 = B$. $\tilde{\triangle}_\ell$ will be referred to as the *cross-over 2-sphere*. If $[\ell]$ is future directed in the future of $\tilde{\triangle}_\ell$, it is past directed in the past. The 1-1 correspondence between functions B and non-extremal $[\ell]$ also implies that, given *any* cross-section $\tilde{\triangle}$, there is a null normal ℓ (defining a non-extremal WIH) which vanishes on it. Thus, *there is a 1-1 correspondence between equivalence classes* $[\ell]$ *of non-extremal WIH null normals and 2-sphere cross-sections* $\tilde{\triangle}_\ell$ *of* $\bar{\triangle}$. We will say that $[\ell]$ and the cross-over 2-sphere $\tilde{\triangle}_\ell$ it defines are *compatible* with one another.

In Section IV B, we introduced a strategy to select a canonical, non-extremal $[\ell]$ on a generic NEH. It is therefore natural to ask if a structure on the associated cross-over 2-sphere can be used to characterize this $[\ell]$. We will now show that the answer is in the affirmative: the expansion of every null vector field n orthogonal to $\tilde{\triangle}_\ell$ *vanishes* on $\tilde{\triangle}_\ell$ identically. (The vanishing is independent of the choice of the null normal n so long as it is non-zero and finite on $\tilde{\triangle}_\ell$.)

To show this, let us suppose that $(\triangle, [\ell])$ is a non-extremal WIH such that $q^{ab}N_{ab} \mathrel{\hat{=}} 0$, i.e., such that for every $n_a = -\mathcal{D}_a v$, with $\mathcal{L}_\ell v \mathrel{\hat{=}} 1$,

$$2\mu \;:\mathrel{\hat{=}}\; \tilde{q}^{ab}\,\mathcal{D}_b n_a$$

is time-independent in the region $\triangle$ of $\bar{\triangle}$ on which ℓ is nowhere vanishing and future directed. Now, since $[\ell]$ vanishes on $\tilde{\triangle}_\ell$, any n satisfying $\ell^a n_a \mathrel{\hat{=}} -1$ must diverge on $\tilde{\triangle}_\ell$. Therefore, to evaluate the transversal expansion of $\tilde{\triangle}_\ell$, let us pass to an appropriately rescaled n, say n^0, which does not diverge or vanish on $\tilde{\triangle}_\ell$. Without loss of the generality, we may assume that ℓ is given by (6.1) with $B = 0$, so we can choose

$$(n^0)_a \;\mathrel{\hat{=}}\; \kappa_{(\ell)} v_0 n_a, \quad \text{or} \quad n_a \;\mathrel{\hat{=}}\; -\frac{1}{\kappa_{(\ell)} v_0}\,\mathcal{D}_a v_0.$$

Then,

$$2\mu_0 :\mathrel{\hat{=}} \tilde{q}_0^{ab}\,\mathcal{D}_a (n^0)_b \mathrel{\hat{=}} \kappa_{(\ell)} v_0 \mu$$

where we have used the fact that $\tilde{q}_0^{ab} \mathrel{\widehat{=}} \tilde{q}^{ab}$ (because n and n^0 are orthogonal to the same family of 2-spheres). Since $(n^0)_a$ is smooth on $\bar{\triangle}$, so is μ_0. But by construction μ is time independent to the future of $\tilde{\triangle}_\ell$ and $v_0 \mathrel{\widehat{=}} 0$ on $\tilde{\triangle}_\ell$. Therefore, we conclude,

$$\mu_0 \mid_{\tilde{\triangle}_\ell} \mathrel{\widehat{=}} 0.$$

Thus, *the preferred cross-section singled out by* $[\ell]$ *is distinguished by the fact that both of its null expansions vanish.* This is an intrinsic property of the cross-section; if it holds for one pair of (well-defined) null normals, it holds for all. [5]

Next, let us consider the converse. Let $\tilde{\triangle}$ be a cross-section of $\bar{\triangle}$ such that the expansion of every null vector field orthogonal to $\tilde{\triangle}$ vanishes everywhere on $\tilde{\triangle}$. We will now show that $[\ell]$ which is compatible with $\tilde{\triangle}$ (i.e., for which $(\triangle, [\ell])$ is a non-extremal WIH and $\ell \mid_{\tilde{\triangle}} = 0$) satisfies $N_{ab}q^{ab} \mathrel{\widehat{=}} 0$. As before, without loss of generality, we may assume, that

$$v_0 \mid_{\tilde{\triangle}} \mathrel{\widehat{=}} 0, \quad \text{and} \quad \ell \mathrel{\widehat{=}} v_0 \ell_0.$$

From (3.12) we know that $(S^0)_{ab} :\mathrel{\widehat{=}} \mathcal{D}_a n_b^0 = -\mathcal{D}_a \mathcal{D}_b v_0$ satisfies

$$\tilde{S}^0_{ab} \mathrel{\widehat{=}} (\tilde{S}^0)^0_{ab} + (\tilde{S}^0)^1_{ab} v_0$$

where $(\tilde{S}^0)^0_{ab}$ and $(\tilde{S}^0)^1_{ab}$ are Lie dragged by both ℓ_0 and ℓ. From the vanishing of the expansion of $(n^0)_a$ on $\tilde{\triangle}$, we conclude

$$\tilde{q}^{ab} (\tilde{S}^0)_{ab} \mid_{v_0=0} \mathrel{\widehat{=}} \tilde{q}^{ab} (\tilde{S}^0)^0_{ab} \mathrel{\widehat{=}} 0.$$

Now, since $v \mathrel{\widehat{=}} \frac{1}{\kappa_{(\ell)}} \ln v_0$ is compatible with ℓ, $\tilde{S}_{ab} \mathrel{\widehat{=}} -\tilde{q}_a{}^c \tilde{q}_b{}^d \mathcal{D}_a \mathcal{D}_b v$ is given just by the rescaling,

$$\tilde{S}_{ab} \mathrel{\widehat{=}} \frac{1}{\kappa_{(\ell)} v_0} (\tilde{S}^0)_{ab} \mathrel{\widehat{=}} \frac{1}{\kappa_{(\ell)} v_0} (\tilde{S}^0)^0_{ab} + \frac{1}{\kappa_{(\ell)}} (\tilde{S}^0)^1_{ab}.$$

Therefore

$$2\mu \mathrel{\widehat{=}} \tilde{q}^{ab} \tilde{S}_{ab} \mathrel{\widehat{=}} \tilde{q}^{ab} (\tilde{S}^0)^1_{ab}, \tag{6.2}$$

is constant along the null generators of $\triangle$.

To summarize, *the cross-over 2-sphere of a* $[\ell]$ *defining a non-extremal WIH is non-expanding in the both orthogonal null directions if and only if* $[\ell]$ *satisfies* $N_{ab}q^{ab} \mathrel{\widehat{=}} 0$. Note that, in this analysis, we did not have to impose the 'genericity' condition of Section IV B: it sufficed to assume that we are given a non-extremal WIH $(\triangle, [\ell])$ on which μ is time-independent.

Finally, let us consider a non-extremal IH $(\triangle, [\ell])$ as in Section V. Repeating the above arguments but using all components of $\tilde{S}_{ab}$ (rather than just the trace) we can conclude that

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[5] Note incidentally that, since n diverges at $\tilde{\triangle}_\ell$, we can not conclude that μ is zero on $\tilde{\triangle}_\ell$: In the (ℓ, n) frame, it is meaningful to calculate spin-coefficients only away from $\tilde{\triangle}$ and μ is time-independent only in that region.

the $\tilde{S}_{ab}$ vanishes on the cross-over 2-sphere $\tilde{\triangle}_\ell$. Conversely, let us suppose that we are given a non-extremal WIH with a cross-section $\tilde{\triangle}$ on which $\tilde{S}_{ab}$ vanishes. Then the $[\ell]$ defined by $\tilde{\triangle}$ endows $\triangle$ with the structure of a non-extremal IH. Thus, on a WIH *the cross-over 2-sphere of* $[\ell]$ *is non-expanding and shear free in the both orthogonal null directions if and only if* $(\triangle, [\ell])$ *is an IH.*

VII. DISCUSSION

In this paper, we analyzed geometrical structures defined intrinsically on non-expanding, weakly isolated and isolated horizons. The intrinsic geometry of a NEH is characterized by the pair $(q, \mathcal{D})$ consisting of a 'metric' q_{ab} of signature (0,+,+) and a compatible derivative operator $\mathcal{D}$. Given *any* null normal ℓ, this pair satisfies two equations: $\mathcal{L}_\ell q_{ab} \,\hat{=}\, 0$ and $\mathcal{D}_a \ell^b \,\hat{=}\, \omega_a \ell^b$ for some 1-form ω on $\triangle$ (see Section II). ω is a potential for the imaginary part of Ψ_2 and determines the angular momentum of $\triangle$ [8]. A WIH is a NEH equipped with an equivalence class $[\ell]$ of null normals such that $\mathcal{L}_\ell \omega \,\hat{=}\, 0$, or, equivalently, $[\mathcal{L}_\ell,\, \mathcal{D}]\ell^a \,\hat{=}\, 0$. (Here two null normals ℓ and ℓ' are regarded as equivalent if they related by a *constant* rescaling.) The notion of 'isolation' provided by this condition suffices to show that the zeroth and the first laws of black hole mechanics can be extended to WIHs [7,8]. On an IH, the normals $[\ell]$ are required to satisfy a stronger condition: $[\mathcal{L}_\ell\,, \mathcal{D}] \,\hat{=}\, 0$.

Thus, as we move from a NEH to an IH, additional intrinsic geometrical structures are assumed to be time-independent. On a NEH the intrinsic metric q_{ab} is time-independent; on a WIH the analog of the 'extrinsic curvature' is also time-independent; while on an IH the entire intrinsic geometry is time-independent. In this sense, the three notions mimic Killing horizons to increasing degrees, thereby capturing the notion that the horizon is 'isolated' in an increasingly stronger sense. Note however that, while every Killing horizon is an IH, the converse is not true. A sub-family of Robinson-Trautman space-times provides explicit examples of space-times which admit isolated horizons but do not admit a Killing vector in any neighborhood of it [23]. More generally, existence theorems [22] based on two intersecting null surfaces have been used to show that Einstein's equations admit an infinite dimensional family of solutions with isolated horizons which are not Killing horizons [21].

We were able to isolate the 'freely specifiable' parts of the intrinsic geometry of WIHs and IHs and show how the remainder is determined by Einstein's equations (possibly with matter sources). We also compared the situation with that at null infinity and with the standard initial value formulation on space-like surfaces (see Sections III B and V). These results clarify the interplay between geometric structures and field equations. A second and perhaps more important set of results concerns the issue of uniqueness of null normals $[\ell]$ which make a given NEH a WIH or an IH. Given a NEH, one can *always* choose a family of null normals $[\ell]$ on it to make it a WIH and furthermore, there is an *infinite*-dimensional freedom in the choice (Section III A). In this sense, the same NEH geometry admits infinitely many WIH structures (Section III A) However, generically, one can select an equivalence class $[\ell]$ uniquely by considering any family of 2-sphere cross-sections of $\triangle$ preserved by the diffeomorphisms generated by ℓ^a and requiring that its transversal expansion should be time-independent. (See Section IV B. In the Newman-Penrose language, the requirement is that the spin-coefficient μ be time-independent.) With IHs, the situation with existence is

quite different: Not every NEH geometry can admit an IH structure. Thus, requiring that a NEH be isolated is a *genuine* restriction. Assuming that the NEH geometry does admit an IH structure, we can ask if the corresponding $[\ell]$ is unique. We showed that for uniqueness to fail the horizon geometry has to be *very* special; in the generic case, the horizon must admit a foliation on which shear and expansion of *both* null normals vanishes and, furthermore, the space-time curvature is severely restricted (see Appendices). In the Kerr family —and hence also in 'nearby solutions'— the uniqueness result does hold. Finally, we also showed that every non-extremal WIH admits an intrinsically defined foliation (Section III C).

These results have significant applications to numerical relativity, particularly to the problem of extracting physics from the numerically evolved strong-field, near-horizon geometry. Consider, for example, dynamical processes in which black holes form or grow due to inflow of radiation and/or matter, or coalescence. The end point of these processes is a single black hole with matter or gravitational radiation in the exterior. Numerical simulations indicate that, at late times, back-scattering becomes negligible and the world tube of apparent horizons becomes a NEH within numerical errors. Then, using the expressions provided by Hamiltonian techniques, one can compute the angular momentum and the mass of these NEHs [7,8], directly in terms of the physical fields *defined at the horizon*. This procedure has already been implemented in numerical codes [2]. Note that this procedure can be carried out without having to embed the given numerical space-time in a probable Kerr geometry, a task which is generally difficult because one has no a priori knowledge of the Kerr metric in the coordinates used in numerical simulations.

Next, using our results from Section IV B, we can generically select an unique equivalence class $[\ell]$ of null normals. Furthermore, using a procedure suggested again by Hamiltonian techniques [7,8], one can even eliminate the freedom to rescale the null normal by a constant and fix the normalization of ℓ entirely in terms of the area and angular momentum of the horizon. In the non-extremal case, generically encountered, one can introduce the geometrical foliation of $\triangle$ (of Section III C) and using transversal geodesics originating from points on these 2-spheres, obtain a foliation of (the near horizon portion of) space-time by a 1-parameter family of null hypersurfaces. These in turn enable one to introduce preferred a null tetrad $(\ell, n, m, \bar{m})$ and coordinates (v, r, θ, ϕ) in the *strong field geometry* near the horizon. Note that the construction is quite rigid: the only freedom is to perform a v, r *independent* $U(1)$ rotation $m \mapsto \exp ifm$ in the tetrad and change coordinates through $v \mapsto v + \text{const}$ and $(\theta, \phi) \mapsto (\theta', \phi')$ where the primed coordinates are independent of v, r. Even this remaining freedom can be eliminated by additional geometrical prescriptions in generic cases. Any geometrical field —such as Ψ_4 in this tetrad— which is insensitive to this freedom is a physical observable. Therefore, it is physically meaningful to directly compare such quantities in *distinct* numerical simulations. In particular, these structures provide a means to meaningfully plot wave forms even in the strong field, near horizon geometry. Next, the past null hypersurface originating in a 2-sphere cross-section in the distant future is likely to be an excellent approximation to future null infinity $\mathcal{I}^+$. Effort is under way to provide expressions of flux of energy and angular momentum carried away by gravitational and other radiation across such null surfaces and analyze their properties. Finally, this framework is also being used to probe the physics of initial data sets. For, if in the binary black hole problem the holes are sufficiently far, one expects from post-Newtonian considerations that the world tubes of the two apparent horizons would be well modelled by

WIHs. The Hamiltonian considerations are then again applicable and, given the full initial data, one can calculate the mass, linear momentum and spin of the two WIHs. Consider for example the Brill-Lindquist [25] initial data for two widely separated black holes so that the distance d between them is much larger than $GM_{\rm ADM}$, where $M_{\rm ADM}$ is the total ADM mass. One finds that the individual horizon masses $M_{\triangle_1}$ and $M_{\triangle_2}$ are related to $M_{\rm ADM}$ via the physically expected relation:

$$M_{\rm ADM} = M_{\triangle_1} + M_{\triangle_2} - G\frac{M_{\triangle_1}M_{\triangle_2}}{d} + \mathcal{O}(\frac{GM_{\rm ADM}}{d})^2.$$

Extension of this relation to more general initial data sets is in progress . This exploration should, in particular, shed some light on the question of 'how much radiation there is' in certain initial data sets.

In the next paper [9], we will use our current results to analyze in detail the *4-dimensional* geometrical structures in space-time regions near weakly isolated and isolated horizons. This analysis paves the way to study perturbations of isolated horizons. A complete characterization of the Kerr isolated horizon [26] already exists. Therefore, study of perturbations will also provide tools to systematically investigate an important issue that has remained largely unexplored: what is the precise sense in which the near horizon geometry approaches that of Kerr space-times in physically interesting dynamical processes?

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APPENDIX A: UNIQUENESS AND EXISTENCE OF IH STRUCTURES

1. Uniqueness

At the end of Section V, we saw that if an NEH geometry $(q, \mathcal{D})$ admits two *non-extremal* IH structures $[\ell]$ and $[\ell']$, then there is a geometry-preserving diffeomorphism on $\triangle$ which maps $[\ell]$ to $[\ell']$. For completeness, we will now address the question: can a NEH geometry support a non-extremal and an extremal IH structure? We will find that this can happen only if that NEH geometry is *very* special.

Recall from Section IV B that if the horizon geometry is generic, i.e., if the elliptic operator $\mathbf{M}$ of equation (4.8) has trivial kernel, the condition $N_{ab}q^{ab} \hat{=} 0$ already implies that $[\ell]$ is unique. In the present case, the burden on the normal is greater; N_{ab} itself has to vanish. Therefore, we will be able to obtain a stronger uniqueness result.

Let $(\triangle, [\ell])$ be an IH. Then $\bar{\ell} \hat{=} \bar{f}\ell$ also satisfies (5.1) if and only if $\bar{N}_{ab} \hat{=} 0$ which, by (4.6), is possible if and only if $\bar{f}$ satisfies:

$$\mathcal{D}_a\mathcal{D}_b\bar{f} + 2\omega_{(a}\mathcal{D}_{b)}\bar{f} \hat{=} 0\,. \tag{A1}$$

As in Section IV B, by transvecting this equation with ℓ^a we conclude that $\bar{f}$ must be of the form (see (4.6))

$$\bar{f} = Be^{-\kappa_{(\ell)} v} + \frac{\kappa_{\bar{\ell}}}{\kappa_{(\ell)}}\,,$$

where $\mathcal{L}_\ell B \,\hat{=}\, 0$ and $\kappa_{\bar{\ell}}$ is the surface gravity of $\bar{\ell}$. Substituting this form back in (A1) we find that on each 2-sphere $\tilde{\triangle}$ defined by $v = \text{const}$, B must satisfy

$$\tilde{\mathcal{D}}_a \tilde{\mathcal{D}}_b B + 2\tilde{\omega}_{(a} \tilde{\mathcal{D}}_{b)} B + \kappa_{(\ell)} \tilde{S}_{ab} B \,\hat{=}\, 0 \tag{A2}$$

where $\tilde{S}_{ab}$ is given in terms of $(\tilde{q}, \tilde{\omega})$ by (5.2). Since the single function B is subject to three different equations on each $\tilde{\triangle}$, a non-trivial solution will exist if and only if the coefficients $(\tilde{q}, \tilde{\omega})$ are severely constrained. Indeed, our discussion in Section IV B shows, generically, there is no non-trivial solution even to the trace of this equation. In the remainder of this sub-section we will explore these constraints under two sets of mild assumptions.

By taking derivatives of this equation one can obtain an integrability condition of the following form

$$\tilde{r}_a{}^b \tilde{\mathcal{D}}_b B \;\hat{=}\; \tilde{s}_a B \tag{A3}$$

with

$$\tilde{r}_a{}^b \;:\;\hat{=}\; 3\tilde{\epsilon}^{cd} (\tilde{\mathcal{D}}_c \tilde{\omega}_d) \tilde{q}_a{}^b + \frac{3}{2} \tilde{\mathcal{R}} \tilde{\epsilon}_a{}^b - R_{cd} \tilde{\epsilon}^{cb} \tilde{q}_a{}^d \tag{A4}$$

$$\tilde{s}_a \;:\;\hat{=}\; 2\kappa_{(\ell)} \tilde{\epsilon}^{dc} (\tilde{\mathcal{D}}_c + \tilde{\omega}_c) \tilde{S}_{da} \;\;\hat{=}\;\; \kappa_{(\ell)} \tilde{\epsilon}^{cd} n_b \tilde{q}_a{}^m R^b{}_{mcd}\,, \tag{A5}$$

where $\tilde{\epsilon}^{ab} = {}^2\epsilon_{cd} \tilde{q}^{ac} \tilde{q}^{bd}$ and $R^b{}_{mcd}$ is the space-time curvature. Let us now assume that $\tilde{r}_a{}^b$ is invertible. If there are no matter fields *on* $\triangle$ this condition is equivalent to assuming that Ψ_2 does not vanish anywhere on $\triangle$, a condition satisfied e.g. by the Kerr family. With this assumption, we can eliminate the derivatives of B in Eq (A2) and obtain

$$B \left[\tilde{\mathcal{D}}_{[a} ((\tilde{r}^{-1})_{b]}{}^c s_c) \right] \;\hat{=}\; 0, \quad \text{and} \tag{A6}$$

$$B \left[(\tilde{r}^{-1})_a{}^c (\tilde{r}^{-1})_b{}^d s_c s_d + \tilde{\mathcal{D}}_{(a} ((\tilde{r}^{-1})_{b)}{}^c s_c) + 2\tilde{\omega}_{(a} (\tilde{r}^{-1})_{b)}{}^c s_c + \kappa_{(\ell)} \tilde{S}_{ab} \right] \;\hat{=}\; 0. \tag{A7}$$

Now, the function B can not be everywhere zero on $\triangle$; if it were, we would have $\bar{\ell} \,\hat{=}\, \text{const}\, \ell$, violating our assumption $[\ell] \not\hat{=} [\bar{\ell}]$. Therefore there is an open region on $\triangle$ on which the terms in square brackets must vanish, thereby constraining the horizon geometry. These are severe constraints and unless they are met the IH horizon structure is unique. In particular, one can show that these conditions can not be met in the Kerr family (and hence in a neighborhood of it in the space of solutions admitting IHs).

We will now explore the constraints imposed by (A2) under a different assumption. Let us suppose that B is nowhere vanishing on $\triangle$ and examine restrictions on the horizon geometry which follow from (A1) and (A2). This is a mild assumption. For example, if the vacuum equations hold on $\triangle$, then the integrability condition (A3) can be solved explicitly and one can show that the solution B can not vanish at any point (see Appendix B). The same conclusion holds also in the case when the only matter field on $\triangle$ is a Maxwell field F_{ab} (such that $\mathcal{L}_\ell F \,\hat{=}\, 0$) [24].

Let us first note that if B satisfies (A2), then $\bar{f} \mathrel{\hat{=}} B\exp(-\kappa_{(\ell)}v) + (\kappa_{\bar{\ell}}/\kappa_{(\ell)})$ as well as $f' \mathrel{\hat{=}} B\exp(-\kappa_{(\ell)}v)$ satisfy (A1). Since by assumption B is nowhere zero, $\ell' \mathrel{\hat{=}} f'\ell$ is a permissible null normal. So, using $\bar{\ell} \mathrel{\hat{=}} \bar{f}\ell$ and $\ell' = f'\ell$, we respectively obtain a non-extremal IH ($IH, [\bar{\ell}]$) *and* an extremal IH $(\triangle, [\ell'])$ on the same horizon geometry. We will now focus on the extremal case and exhibit the stringent restrictions on the horizon geometry imposed by the assumption that the IH structure is not unique.

It is straightforward to verify that $v' :\mathrel{\hat{=}} (1/\kappa_{(\ell)}B)e^{\kappa_{(\ell)}v}$ is an adapted coordinate for ℓ'. Set $n' \mathrel{\hat{=}} -dv'$. This n' satisfies our equations (3.5) for ℓ' and we will denote by $\tilde{\triangle}'$, the 2-spheres $v' = \text{const}$ orthogonal to it. Now,

$$\frac{1}{v'}\, n'_b \mathrel{\hat{=}} \kappa_{(\ell)}n_b + \mathcal{D}_b \ln B$$

and taking derivatives of both sides we obtain

$$\frac{1}{v'}\, S'_{ab} + \frac{1}{v'^2}\, n'_a n'_b \mathrel{\hat{=}} \kappa_{(\ell)}S_{ab} + \mathcal{D}_a\mathcal{D}_b B.$$

Let us now use the fact that $N_{ab} \mathrel{\hat{=}} \mathcal{L}_\ell S_{ab} \mathrel{\hat{=}} 0$ and $N'_{ab} = \mathcal{L}_{\ell'} S'_{ab} \mathrel{\hat{=}} 0$. This implies

$$S'_{ab} \mathrel{\hat{=}} -2\omega'_{(a}n'_{b)}.$$

Transvecting this equation with ℓ', we only obtain the identity $S'_{ab}\ell'^b \mathrel{\hat{=}} \omega'_{\underset{\sim}{a}}$. But the pull-back of the equation to the 2-spheres $\tilde{\triangle}'$ yields

$$\tilde{S}'_{ab} \mathrel{\hat{=}} 0\,.$$

In the Newman-Penrose notation of Appendix B, we conclude that $\rho', \sigma', \kappa_{(\ell')}, \mu', \lambda'$ all vanish and m_a, π' are constrained by (5.3). [6] This is a very strong restriction on the horizon geometry. For, it implies that: i) the horizon can be foliated by a family of 2-spheres $\tilde{\triangle}$ for which expansions and shears of *both* families of orthogonal null normals vanish; and, ii) $(q, \tilde{\omega})$ are severely constrained by (5.3).

To summarize, a NEH geometry admitting two inequivalent IH structures is severely restricted. Under two different sets of mild assumptions we exhibited these restrictions explicitly. The second set has a simple geometrical interpretation.

2. Existence of an IH structure.

As noted in Section V, not all NEHs admit an IH structure, i.e., a null normal ℓ such that $[\mathcal{L}_\ell, \mathcal{D}] \mathrel{\hat{=}} 0$. In this sub-section, we will exhibit some conditions that the geometry $(q, \mathcal{D})$ of an NEH must satisfy for such an ℓ to exist. Although the geometrical meaning of these conditions is not transparent, they serve to bring out the non-triviality of the passage

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[6] If the only matter field on $\triangle$ is a Maxwell field, one can show that these conditions imply that (q, ω') are necessarily those of the extremal Kerr-Newman space-time [24].

from WIHs to IHs. They also provide a practical tool to show that a given NEH does not admit any IH structure.

Let $\triangle$ be an NEH. To determine if it admits an IH structure, as in Section IV B, one can just construct the $[\ell]$ which endows $\triangle$ with a preferred WIH structure and ask if it satisfies (5.1). However, this criterion is often not useful in practice because, given a specific WIH, it may not be possible to find explicitly the canonical $[\ell]$ required in this construction. In this sub-section, we will derive a set of conditions which must be satisfied by *any* non-extremal WIH structure $(\triangle, [\ell], q, \mathcal{D})$ if there is to exist *some* null normal ℓ' such that $(\triangle, [\ell'], \mathcal{D})$ satisfies (5.1). (See Appendix B for further details in the N-P notation).

Let $(\triangle, [\ell])$ be a WIH of $\kappa_{(\ell)} \not\cong 0$ and suppose the geometry of $\triangle$ admits an non-extremal IH $[\ell']$. As before,

$$\ell' \ \hat{=}\ (Be^{-\kappa_{(\ell)}v} + \frac{\kappa_{(\ell')}}{\kappa_{(\ell)}})\ell, \tag{A8}$$

where $\mathcal{L}_\ell B \hat{=} 0$ and v is compatible with ℓ. It follows from (4.5) and $[\mathcal{L}_{\ell'}, \mathcal{D}_a] \hat{=} 0$, that B satisfies the following system of equations

$$\Big[\tilde{\mathcal{D}}_a\tilde{\mathcal{D}}_b + 2\tilde{\omega}_{(a}\tilde{\mathcal{D}}_{b)} + \kappa_{(\ell)}\tilde{S}^1_{ab}\Big]B - \kappa_{(\ell')}\tilde{S}^0_{ab} \hat{=} 0, \tag{A9}$$

where $\tilde{S}_{ab} \hat{=} \tilde{S}^0_{ab}e^{\kappa_{(\ell)}v} + \tilde{S}^1_{ab}$, and $\tilde{S}^0_{ab}$ and $\tilde{S}^1_{ab}$ are Lie dragged by ℓ and given by (3.11). Again, since these are three differential equations on a single function B, a non-trivial solution can exist only if the coefficients are suitably constrained.

An integrability condition can be derived by acting on (A9) by $\tilde{\mathcal{D}}_c$:

$$\tilde{r}_a{}^b\tilde{\mathcal{D}}_bB \ \hat{=}\ \tilde{s}^1_aB - \tilde{s}^0_a \tag{A10}$$

where $\tilde{r}_a{}^b$ and $\tilde{s}_a$ are defined in (A4) and (A5), and $\tilde{s}^0_ae^{\kappa_{(\ell)}v}\ + \tilde{s}^1_a\ \ :\hat{=}\ \tilde{s}_a$. Substituting for $\tilde{\mathcal{D}}_aB$ in (A9) we obtain two necessary conditions in which B appears only algebraically,

$$B\Big[\tilde{\mathcal{D}}_{[a}(\tilde{r}^{-1})_{b]}{}^c\tilde{s}^1_c\Big] - \tilde{\mathcal{D}}_{[a}(\tilde{r}^{-1})_{b]}{}^c\tilde{s}^0_c) + (\tilde{r}^{-1})_a{}^c(\tilde{r}^{-1})_b{}^d s^1_{[c}s^0_{d]} \ \hat{=}\ 0, \tag{A11}$$

$$\begin{aligned} B\Big[(\tilde{r}^{-1})_a{}^c(\tilde{r}^{-1})_b{}^d\tilde{s}_c\tilde{s}_d + \tilde{\mathcal{D}}_{(a}((\tilde{r}^{-1})_{b)}{}^c\tilde{s}^1_c) + 2\tilde{\omega}_{(a}(\tilde{r}^{-1})_{b)}{}^c\tilde{s}^1_c + \kappa_{(\ell)}\tilde{S}^1_{ab}\Big] - \\ \tilde{\mathcal{D}}_{(a}((\tilde{r}^{-1})_{b)}{}^c\tilde{s}^0_c) - 2\tilde{\omega}_{(a}(\tilde{r}^{-1})_{b)}{}^c s^0_c + (\tilde{r}^{-1})_{(a}{}^c(\tilde{r}^{-1})_{b)}{}^d\tilde{s}^1_c\tilde{s}^0_d - \kappa_{(\ell')}S^0_{ab} \ \hat{=}\ 0. \end{aligned} \tag{A12}$$

Thus, if an IH structure is to exist, these equations must admit a solution for a nowhere vanishing function B. Since each equation is of the type $B\alpha + \beta \hat{=} 0$ where the coefficients α and β depend only on the geometry $(q, \mathcal{D})$, the geometry $(q, \mathcal{D})$ of the IH is constrained.

APPENDIX B: MAIN RESULTS IN THE NEWMAN-PENROSE NOTATION

Since the Newman-Penrose framework [16] is geared to null surfaces, it is well-suited for detailed calculations involving the three types of horizons considered in this paper. We chose not to use it in the main body of the paper only because the structures of interest refer only to a null vector ℓ^a rather than to a full null tetrad, whence expansions of geometric fields in null tetrads can obscure the underlying covariance. However, for the convenience of readers

more familiar with the Newman-Penrose framework, in this appendix we will translate our main results to that notation.

To prevent a proliferation of symbols, we will use the same symbol to denote co-vectors and their pullbacks from the space-time onto $\triangle$; the context will make it clear which of the two possibilities is intended.

1. Null surface geometry

Let us begin with a general null surface $\triangle$, not necessarily a non-expanding horizon (NEH). A quadruple of vectors $(m^a, \bar{m}^a, n^a, \ell^a)$ will be said to be a *null tetrad* if the only non-vanishing scalar products of its elements are

$$m^a \bar{m}_a \;=\; 1 \;=\; -\ell^a n_a,$$

where, m is complex valued and n, ℓ are real. Following the Newman-Penrose notation, we will denote the directional derivatives along null tetrads by

$$\delta = m^a \partial_a, \quad D = \ell^a \partial_a, \quad \Delta = \Delta^a \partial_a. \tag{B1}$$

The dual co-frame is given by $(\bar{m}_a, m_a, -\ell_a, -n_a)$. We will assume that the vector field ℓ^a is tangent to $\triangle$. It then follows that $\mathrm{Re} m^a$ and $\mathrm{Im} m^a$ are also tangential and the pullback of ℓ_a to $\triangle$ vanishes. The vectors $(m^a, \bar{m}^a, \ell^a)$ span the tangent space to $\triangle$ while the dual co-frame is given by the pullbacks of $(\bar{m}_a, m_a, -n_a)$.

In terms of the null tetrad, the degenerate metric tensor induced on $\triangle$, is given by

$$q_{ab} \;=\; m_a \bar{m}_b + \bar{m}_a m_b, \tag{B2}$$

2. Non-expanding horizons

On a non-expanding horizon, the expansion and shear of ℓ (i.e., the Newman-Penrose spin coefficients ρ and σ) vanish. As a result, as explained in Section II A, the space-time derivative operator ∇ compatible with the 4-metric g induces an intrinsic covariant derivative operator $\mathcal{D}$ on $\triangle$. Being intrinsic to $\triangle$, it is completely defined by its action on ℓ^a, n_a and m_a. In the Newman-Penrose framework this action can be expressed explicitly as:

$$\mathcal{D}_a \ell^b \,\hat{=}\, \omega_a \ell^b \tag{B3}$$

$$\bar{m}^b \nabla_a n_b \,\hat{=}\, \lambda m_a + \mu \bar{m}_a - \pi n_a \tag{B4}$$

$$m^b \nabla_a \bar{m}_b \,\hat{=}\, -(\alpha - \bar{\beta}) m_a + (\bar{\alpha} - \beta) \bar{m}_a + (\epsilon - \bar{\epsilon}) n_a \,\hat{=}\, -\bar{m}^b \mathcal{D}_a m_b\,. \tag{B5}$$

Here, all spin coefficients are complex and the 1-form ω_a is expressed in terms of them via

$$\omega_a \;\hat{=}\; (\alpha + \bar{\beta}) m_a + (\bar{\alpha} + \beta) \bar{m}_a - (\epsilon + \bar{\epsilon}) n_a, \tag{B6}$$

The fact that $\mathcal{D}$ is compatible with q_{ab} follows from the fact that $m^b \mathcal{D}_a \bar{m}_b$ is imaginary, while the torsion-free property, $\mathcal{D}_{[a} \mathcal{D}_{b]} f \,\hat{=}\, 0$, of $\mathcal{D}$ can expressed via

$$\delta\bar{\delta} - \bar{\delta}\delta \;\hat{=}\; (\mu - \bar{\mu}) D - (\alpha - \bar{\beta}) \delta + (\bar{\alpha} - \beta) \bar{\delta} \tag{B7}$$

$$\delta D - D\delta \;\hat{=}\; (\bar{\alpha} + \beta - \bar{\pi}) D - (\epsilon - \bar{\epsilon}) \delta \tag{B8}$$

a. Constraint Equations

The pullback $R_{\underleftarrow{ab}}$ of the space-time Ricci tensor to $\triangle$ is completely determined by q and $\mathcal{D}$:

$$\Phi_{00} := \frac{1}{2} R_{\ell\ell} \;\hat{=}\; 0 \tag{B9}$$

$$\Phi_{10} := \frac{1}{2} R_{\ell\bar{m}} \;\hat{=}\; D\alpha - \bar{\delta}\epsilon - (\bar{\epsilon} - 2\epsilon)\alpha + \epsilon\bar{\beta} - (\epsilon)\pi \tag{B10}$$

$$\Phi_{20} := \frac{1}{2} R_{\bar{m}\bar{m}} \;\hat{=}\; D\lambda - \bar{\delta}\pi - \pi^2 - (\alpha - \bar{\beta})\pi + (3\epsilon - \bar{\epsilon})\lambda \tag{B11}$$

$$\begin{aligned}\Phi_{11} + \frac{1}{8}R := \frac{1}{2} R_{m\bar{m}} \;\hat{=}\;& D\mu - \delta\pi - \pi\bar{\pi} + (\epsilon + \bar{\epsilon})\mu + (\bar{\alpha} - \beta)\pi \\ &+ \delta\alpha - \bar{\delta}\beta - \alpha\bar{\alpha} - \beta\bar{\beta} + 2\alpha\beta - \epsilon(\mu - \bar{\mu})\end{aligned} \tag{B12}$$

where R is the space-time Ricci scalar.

The energy condition iii) in Definition 1 of a NEH implies $R_{ab}\ell^a\ell^b \;\hat{=}\; 0$ and $R_{ab}\ell^a\bar{m}^b \;\hat{=}\; 0$. The first of these was ensured in (B9) by the fact that the shear and divergence of ℓ vanish on $\triangle$. The second equation imposes restrictions on spin-coefficients. To simplify it, let us choose the tetrad vector m^a such that

$$\epsilon \;\hat{=}\; \bar{\epsilon}, \tag{B13}$$

This 'gauge choice' can *always* be made and *we will employ it from now on.* Then, using the torsion-free conditions (B7) and (B8) satisfied by $\mathcal{D}$, the second restriction, $R_{ab}\ell\bar{m} \;\hat{=}\; 0$ can be expressed as:

$$D(\alpha + \bar{\beta}) - 2\bar{\delta}\epsilon \;\hat{=}\; 2\epsilon(\pi - \alpha - \bar{\beta}). \tag{B14}$$

3. Weakly isolated horizons

Let us now assume that ℓ is so chosen that $(\triangle, [\ell])$ is a weakly isolated horizon (WIH). Then, it immediately follows that [7]

$$\kappa_{(\ell)} := 2\epsilon \;\hat{=}\; \text{const.} \tag{B15}$$

Next, let us choose n as follows:

$$n \;\hat{=}\; -\, dv, \quad \text{where} \quad Dv \;\hat{=}\; 1.$$

[7] The only departure from the standard Newman-Penrose notation we make is to denote surface gravity $(\epsilon + \bar{\epsilon})$ by $\kappa_{(\ell)}$, following the convention of black hole mechanics. This should not cause any confusion because the fact that ℓ is necessarily geodesic on a null surface implies that the Newman-Penrose κ vanishes identically.

In this gauge

$$\mu \;\hat{=}\; \bar{\mu}, \quad \pi \;\hat{=}\; \alpha + \bar{\beta}. \tag{B16}$$

The projection operator onto the leaves of the foliation $v \,\hat{=}\, \text{const}$, can be expressed as

$$\tilde{q}^a{}_b \;\hat{=}\; m_b \bar{m}^a + \bar{m}_b m^a. \tag{B17}$$

and the projected fields $\tilde{\omega}_a$ and $\tilde{S}_{ab}$ of Section III B can be expressed as

$$\tilde{\omega}_a \;:\hat{=}\; \tilde{q}^b{}_a \omega_b \;\hat{=}\; (\alpha + \bar{\beta}) m_a + (\bar{\alpha} + \beta)\bar{m}_a \tag{B18}$$

$$\tilde{S}_{ab} :\hat{=} \tilde{q}^c_a \tilde{q}^d_b \mathcal{D}_c n_d \;\hat{=} \mu(m_a \bar{m}_b + \bar{m}_a m_b) + \lambda m_a m_b + \bar{\lambda} \bar{m}_a \bar{m}_b. \tag{B19}$$

a. Constraints (B11) and (B12)

Since the pull-backs of the space-time Ricci tensor to $\triangle$ is determined completely by the geometry $(q, \mathcal{D})$ of the WIH, via field equations the stress-energy tensor at the horizon constrains this geometry. (The other components of the Ricci tensor involve new information, not contained in $(q, \mathcal{D})$ and therefore do not impose any such restrictions.) We have already analyzed the consequences of (B9) and (B10). We will now analyze the constraints imposed by the remaining two equations. They determine the evolution of $\tilde{S}_{ab}$ along the null normal ℓ^a:

$$\mathcal{L}_\ell \tilde{S}_{ab} \;\hat{=} (D\mu)\,(m_a \bar{m}_b + \bar{m}_a m_b) + (D\lambda)\, m_a m_b + (D\bar{\lambda})\, \bar{m}_a \bar{m}_b. \tag{B20}$$

Because ℓ is also tangential to $\triangle$, this 'evolution' equation is in fact a constraint. By decomposing this equation in to various components, we obtain:

$$D\mu \,\hat{=}\, -\kappa_{(\ell)}\mu + \frac{1}{2}\Big(\hat{\text{div}}\tilde{\omega} + 2\pi\bar{\pi} - K + 2\Phi_{11} + \frac{1}{4}R\Big), \tag{B21}$$

$$D\lambda \,\hat{=}\, -\kappa_{(\ell)}\lambda + \bar{\delta}\pi + (\alpha - \bar{\beta})\pi + \pi^2 + \Phi_{20}, \tag{B22}$$

$$\tag{B23}$$

where,

$$K \,\hat{=}\, \delta(\alpha - \bar{\beta}) + \bar{\delta}(\bar{\alpha} - \beta) - 2(\alpha - \bar{\beta})(\bar{\alpha} - \beta), \tag{B24}$$

$$\hat{\text{div}}\hat{\omega} \,\hat{=}\, \delta(\pi) + \bar{\delta}(\bar{\pi}) - (\alpha - \bar{\beta})\bar{\pi} - (\bar{\alpha} - \beta)\pi. \tag{B25}$$

$$\tag{B26}$$

(K is the Gauss curvature of $\tilde{q}_{ab}$). Finally, in the Newman-Penrose notation, the remaining components of the pullback $R^a{}_{\underleftarrow{bcd}}$ of the space-time Riemann tensor onto $\triangle$ are given by

$$\Psi_0 \;\hat{=}\; 0, \quad \Psi_1 \;\hat{=}\; 0 \tag{B27}$$

$$\Psi_2 + \frac{R}{12} \;\hat{=}\; D\mu + \kappa_{(\ell)}\mu - \delta\pi + (\bar{\alpha} - \beta)\pi - \pi\bar{\pi} \tag{B28}$$

$$\Psi_3 \;-\; \Phi_{21} \;\hat{=}\; \bar{\delta}\mu - \delta\lambda + \pi\mu + \lambda(\bar{a} - 3\beta). \tag{B29}$$

where

$$\Psi_0 \;=\; C_{\alpha\beta\gamma\delta}\ell^\alpha m^\beta \ell^\gamma m^\delta, \quad \Psi_1 \;=\; C_{\alpha\beta\gamma\delta}\ell^\alpha n^\beta \ell^\gamma m^\delta, \tag{B30}$$

$$\Psi_2 \;=\; \frac{1}{2} C_{\alpha\beta\gamma\delta}\ell^\alpha n^\beta (\ell^\gamma n^\delta - m\bar{m}), \tag{B31}$$

$$\Psi_3 \;=\; C_{\alpha\beta\gamma\delta} n^\alpha \ell^\beta n^\gamma \bar{m}^\delta, \quad \Phi_{21} \;=\; \frac{1}{2} R_{ab} n^a \bar{m}^b. \tag{B32}$$

b. Good cuts and the canonical WIH structures

Given a non-extremal WIH $(\triangle, [\ell])$, a representative null normal ℓ, as shown in Section III C we can obtain a preferred foliation of $\triangle$. The leaves of this foliation are called *good cuts* of $\triangle$. Let us we label these cuts by $v = \text{const}$ with $Dv \;\hat{=}\; 1$ and set $n = -dv$. Then, in the Newman-Penrose notation, these cuts are characterized by the following equations:

$$\hat{\text{div}}\hat{\omega} \;\hat{=}\; \delta(\pi) + \bar{\delta}(\bar{\pi}) - (\alpha - \bar{\beta})\bar{\pi} - (\bar{\alpha} - \beta)\pi \;\hat{=}\; 0 \quad \text{and} \quad \pi \;\hat{=}\; -i\bar{\delta}U. \tag{B33}$$

Next, given a NEH $\triangle$, the strategy of Section IV B is to select a canonical $[\ell]$ by requiring that $(\triangle, [\ell])$ be a WIH satisfying

$$D\mu \;\hat{=}\; 0. \tag{B34}$$

Our main result of Section IV B can be stated as follows: if the operator

$$\mathbf{M} \;\hat{=}\; \delta\bar{\delta} + \bar{\delta}\delta - (\alpha - \bar{\beta})\delta - (\bar{\alpha} - \beta)\bar{\delta} + \hat{\text{div}}\hat{\omega} + 2\pi\delta + 2\bar{\pi}\bar{\delta} + 2\pi\bar{\pi} - K + 2\Phi_{11} + \frac{1}{4}R, \tag{B35}$$

has a trivial kernel, then the canonical $[\ell]$ exists, is non-extremal and unique. In this case, μ is determined by other horizon fields via:

$$\mu \;\hat{=}\; \frac{1}{2\kappa_{(\ell)}}\Big(\hat{\text{div}}\tilde{\omega} + 2\pi\bar{\pi} - K + 2\Phi_{11} + \frac{1}{4}R\Big). \tag{B36}$$

4. Implications of non-unique IH structures

a. The general case

Let $(\triangle, [\ell])$ be an IH. We now assume that the underlying horizon geometry $(q, \mathcal{D})$ admits a distinct IH structure $[\ell']$ and investigate the consequences of this non-uniqueness. In this case have:

$$\ell' \;=\; f\ell, \tag{B37}$$

and using only the WIH properties of ℓ and ℓ' we know that,

$$f \;\widehat{=}\; Be^{-\kappa_{(\ell)}v} + \frac{\kappa_{(\ell')}}{\kappa_{(\ell)}} \text{ if } \kappa_{(\ell)} \not\widehat{=} 0 \tag{B38}$$

$$\text{and } \; f \;\widehat{=}\; \kappa_{(\ell')}v - B \text{ if } \kappa_{(\ell)} = 0, \tag{B39}$$

where $Dv \;\widehat{=}\; 1$ and $DB \;\widehat{=}\; 0$. Let us first consider the case when $\kappa_{(\ell)} \not\widehat{=} 0$. Then, the condition $[\mathcal{L}_\ell, \mathcal{D}] \;\widehat{=}\; 0 \;\widehat{=}\; [\mathcal{L}_{\ell'}, \mathcal{D}]$ is equivalent to the following set of equations on a cross-section $\tilde{\triangle}$ of $\triangle$:

$$\Big[\frac{1}{2}(\delta\bar\delta + \bar\delta\delta - (\alpha - \bar\beta)\delta - (\bar\alpha - \beta)\bar\delta) + \pi\delta + \bar\pi\bar\delta + \mu\kappa_{(\ell)}\Big]B \;\widehat{=}\; 0 \tag{B40}$$

$$\Big[(\delta + \bar\alpha - \beta + 2\bar\pi)\delta + \bar\lambda\kappa_{(\ell)}\Big]B \;=\; 0. \tag{B41}$$

on a cross-section $\tilde{\triangle}$ of $\triangle$. These are equivalent to Eq (A2) in the main text

The integrability conditions of this set are given by Eq (A3), namely $\tilde{r}_a{}^b\mathcal{D}_bB \;\widehat{=}\; \tilde{s}_aB$. In the Newman-Penrose notation, we have

$$\tilde{r}_a{}^b \;\widehat{=}\; 2i\Big[\Phi_{20}m_am^b - \Phi_{02}\bar{m}_a\bar{m}^b - (3\Psi_2 - 2\Phi_{11})m_a\bar{m}^b + (3\overline{\Psi_2} - 2\Phi_{11})\bar{m}_am^b\Big] \tag{B42}$$

$$\tilde{s}_a \;\widehat{=}\; 2i\Big[(\Psi_3 - \Phi_{21})m_a - (\overline{\Psi_3} - \Phi_{12})\bar{m}_a\Big]. \tag{B43}$$

and (A3) reduces to:

$$(3\Psi_2 \; - \; 2\Phi_{11})\bar\delta B - \Phi_{20}\delta B \;=\; -\kappa_{(\ell)}(\Psi_3 - \Phi_{21})B. \tag{B44}$$

In the main text we used the inverse of $(\tilde{r}^{-1})_a{}^b$ to express $\mathcal{D}_aB$ in terms of B and the horizon geometry. This expression simplifies if we make a mild assumption on matter fields on $\triangle$, namely,

$$\Phi_{20} \;\widehat{=}\; 0, \tag{B45}$$

which automatically holds, in particular, in the electrovac case. Then the matrix $\tilde{r}_a{}^b$ is diagonal in the null frame, and invertible at any point of $\triangle$ at which

$$3\Psi_2 - 2\Phi_{11} \;\not\widehat{=}\; 0. \tag{B46}$$

At these points, $(\tilde{r}^{-1})_a{}^b\tilde{s}_b = \bar\Psi m_a + \Psi\bar{m}_a$ where

$$\bar\Psi \;:\widehat{=}\; -\frac{\Psi_3 - \Phi_{21}}{3\Psi_2 - 2\Phi_{11}}. \tag{B47}$$

Thus, if $\Phi_{20} \;\widehat{=}\; 0$, the restriction that $\tilde{r}_a{}^b(x)$ be invertible reduces to the condition that $3\Psi - 2\Phi_{11}$ be non-zero at x. On the part of $\triangle$ on which Ψ is well-defined, the integrability conditions imply

$$B\,\mathrm{Im}\,(\delta + \beta - \bar\alpha)\bar\Psi \;\widehat{=}\; 0. \tag{B48}$$

Now, B can not vanish identically; if it did, $\ell' = \mathrm{const}\,\ell$, contradicting our assumption $[\ell] \not\widehat{=} [\ell']$. On the portion of $\triangle$ where B does not vanish, the horizon geometry is constrained.

So far we have focussed on the integrability conditions for (B40) and (B41). These latter equations themselves impose further constraints on the horizon geometry. Using the integrability conditions to substitute for derivatives of B in terms of B in these equations, we obtain:

$$B\Big[(\bar\delta + \bar\beta - \alpha)\Psi \ + \pi\Psi + \bar\pi\bar\Psi + \kappa_{(\ell)}\Psi\bar\Psi + \mu\Big] \ \hat{=}\ 0, \tag{B49}$$

$$\text{and} \quad B\Big[(\delta + \bar\alpha - \beta)\Psi \ + 2\bar\pi\Psi + \kappa_{(\ell)}\Psi^2 + \bar\lambda\Big] \ \hat{=}\ 0 \tag{B50}$$

Since B can not vanish identically, the horizon geometry which (together with the tetrad) determines the terms in the square brackets is constrained.

Finally, let us consider the case when $\kappa_{(\ell)} \hat{=} 0$. Then, (B40) and (B41) are replaced by

$$\Big[\frac{1}{2}\delta\bar\delta + \bar\delta\delta - (\alpha - \bar\beta)\delta - (\bar\alpha - \beta)\bar\delta + \pi\delta + \bar\pi\bar\delta\Big]B \ \hat{=}\ -\mu\kappa_{(\ell')} \tag{B51}$$

$$\text{and} \quad \Big[(\delta + \bar\alpha - \beta + 2\bar\pi)\delta\Big]B \,\hat{=}\, -\bar\lambda\kappa_{(\ell')}; \tag{B52}$$

and the integrability conditions (B53) are replaced by

$$(3\Psi_2 \ - \ 2\Phi_{11})\bar\delta\, B - \Phi_{20}\delta\, B \ \hat{=}\ -\kappa_{(\ell')}(\Psi_3 - \Phi_{21})B. \tag{B53}$$

Assuming $\Phi_{20} \hat{=} 0$ and Ψ is well-defined, B again satisfies (B48) if $\kappa_{(\ell')} \hat{\neq} 0$, but

$$B \,\hat{=}\, \text{const} \tag{B54}$$

if $\kappa_{(\ell')} \hat{=} 0$. Finally, if $\kappa_{(\ell')} \neq 0$ equations (B51) and(B52) imply

$$B\mathrm{Re}\Big[(\delta + \beta - \bar\alpha + \kappa_{(\ell')}\Psi + 2\bar\pi)\bar\Psi\Big] + \mu \ \hat{=}\ 0 \tag{B55}$$

$$B\Big[(\delta + \bar\alpha - \beta + 2\bar\pi + \kappa_{(\ell')}\Psi)\Psi\Big] + \bar\lambda \ \hat{=}\ 0 \tag{B56}$$

$$B\mathrm{Im}\Big[\delta + \beta - \bar\alpha\Big]\,\bar\Psi \,\hat{=}\, 0. \tag{B57}$$

To summarize, if the IH horizon structure is not unique, the horizon geometry is constrained both in the non-extremal and extremal cases. We exhibited these constraints under the assumption that $\Phi_{20} \hat{=} 0$ and Ψ is well-defined.

b. Simplifications in the non-extremal, vacuum case.

Let us now suppose $R_{\underleftarrow{ab}} \hat{=} 0$ and $\kappa_{(\ell)} \hat{\neq} 0$ on $\triangle$. We will now show that in this case B and Ψ_2 can not vanish anywhere (whence Ψ is well-defined everywhere) on $\triangle$. Under our present assumptions, Bianchi identities yield:

$$\kappa_{(\ell)}\Psi_3 \ \hat{=}\ (\bar\delta + 3\pi)\Psi_2. \tag{B58}$$

The integrability condition (B44) is equivalent to

$$B^3\Psi_2\, e^{-3iU} \ \hat{=}\ C \tag{B59}$$

where C is a constant. If $C \not\triangleq 0$, B and Ψ_2 must be everywhere non-vanishing as we wished to show. Let us therefore consider the other case, $C = 0$. Now, B can not vanish identically; if it did $[\ell] = [\ell']$, contradicting our initial assumption. If $C = 0$, Eq (B59) implies that Ψ_2 must vanish on the region on which B is non-zero and Eq (B40) implies

$$\hat{\Delta} B \;=\; 0, \quad \text{whenever} \quad \Psi_2 \;=\; 0. \tag{B60}$$

Let $\bar{S}$ be the closure of the support of B. Since $\hat{\Delta} B = 0$ on $\bar{S}$ and B vanishes on the boundary of $\bar{S}$, B must vanish on $\bar{S}$. This implies B vanishes everywhere on $\hat{\triangle}$, contradicting our assumption. Thus, the constant C can not be zero.

To summarize if $\kappa_{(\ell)} \not\triangleq 0$ and $R_{\underleftarrow{ab}} \triangleq 0$, then the assumptions $\Phi_{20} \triangleq 0$ of the last subsection is trivially satisfied and, furthermore, Ψ is well-defined globally on $\triangle$. In this case, the geometry is severely constrained and, if it exists, B is given by (B59).

c. The non-extremal, vacuum, non-rotating case.

Let as apply the above results of the last subsection to non-rotating horizons, i.e., horizons satisfying

$$\tilde{\omega}_a \;\triangleq\; 0. \tag{B61}$$

Note that the intrinsic metric of these horizons need not be spherical; arbitrary distortions are permissible. In this case, $-\Psi_2 = K$ whence (B59) implies

$$B \;\triangleq\; B_0 K^{-\frac{1}{3}} \tag{B62}$$

where B_0 is a constant. Integrating equation (B40) on $\hat{\triangle}$ and substituting for μ from (5.2) we conclude

$$(\hat{\Delta} - K) K^{-\frac{1}{3}} \triangleq 0. \tag{B63}$$

Finally, using the fact that the image of the Laplace operator is orthogonal to the constant function, we find

$$0 \;\triangleq\; B_0 \int_{\hat{\triangle}} (\hat{\Delta} - K) K^{-\frac{1}{3}} \hat{\epsilon} \;=\; -B_0 \int_{\hat{\triangle}} (K^{\frac{1}{3}})^2 \hat{\epsilon} \tag{B64}$$

Now, because of our assumptions K is nowhere vanishing, whence the integral is positive definite. This implies $B_0 \triangleq 0$ and therefore $[\ell] \triangleq [\ell']$. Thus, in the non-extremal, non-rotating, vacuum case, we conclude that *if a NEH admits an IH structure, that structure is unique.*

5. The existence conditions.

Finally, we will recast the discussion of Appendix A 2 in the Newman-Penrose language. Let $\triangle$ be a NEH. Choose *any* null normal ℓ which endows it the structure of a non-extremal,

weakly isolated horizon and denote by v a compatible coordinate. Suppose ℓ' is another null normal which defines an IH structure on $\triangle$. Then, $\ell' = Be^{-\kappa_{(\ell)}v} + \frac{\kappa_{(\ell')}}{\kappa_{(\ell)}}$ (with $DB = 0$) and the geometry satisfies the conditions (A9). In the Newman-Penrose notation, they read,

$$\frac{1}{2}\Big[\bar{\delta}\delta + \delta\bar{\delta} + (2\pi - \alpha + \bar{\beta})\delta + (2\bar{\pi} - \bar{\alpha} + \beta)\bar{\delta} + 2\kappa_{(\ell)}\mu^1\Big]B \;\hat{=}\; \kappa_{(\ell')}\mu^0 \tag{B65}$$

$$\text{and} \quad \Big[\bar{\delta}\bar{\delta} + (2\pi + \alpha - \bar{\beta})\bar{\delta} + \kappa_{(\ell)}\lambda^1\Big]B \;\hat{=}\; \kappa_{(\ell')}\lambda^0, \tag{B66}$$

where $\mu \;\hat{=}\!:\; \mu^0 e^{-\kappa_{(\ell)}v} + \mu^1$ and $\lambda \;\hat{=}\!:\; \lambda^0 e^{-\kappa_{(\ell)}v} + \lambda^1$ with $D\mu^0 \hat{=} D\mu^1 \hat{=} D\lambda^0 \hat{=} D\lambda^1 \hat{=} 0$. The $\tilde{r}_a{}^b$ and $\tilde{s}_a$ used in the integrability condition (A10) are expressed in the Newman-Penrose formalism in Eqs (B42) and (B43). Using these expressions, the integrability condition reads

$$(3\Psi_2 \;-\; 2\Phi_{11})\bar{\delta}B - \Phi_{20}\delta B \;\hat{=}\; -\kappa_{(\ell)}(\Psi_3 - \Phi_{21})^1 B + \kappa_{(\ell')}(\Psi_3 - \Phi_{21})^0, \tag{B67}$$

$$\text{and} \quad (\Psi_3 - \Phi_{21}) \;\hat{=}\!:\; (\Psi_3 - \Phi_{21})^0 e^{-\kappa_{(\ell)}v} + (\Psi_3 - \Phi_{21})^1, \tag{B68}$$

where $D(\Psi_3 - \Phi_{21})^0 \hat{=} D(\Psi_3 - \Phi_{21})^1 \hat{=} 0$. Let us write down the equations (A12) and (A11) assuming again $\Phi_{02} \;\hat{=}\; 0$, thereby making the matrix $\tilde{r}_a{}^b$ diagonal in the null frame. The function Ψ of (B47) also has the form

$$\Psi \;:\hat{=}\; \Psi^0 e^{-\kappa_{(\ell)}v} + \Psi^1, \tag{B69}$$

where $D\Psi^0 \;\hat{=}\; D\Psi^1 \;\;\hat{=}\; 0$. Then, wherever $3\Psi_2 - 2\Psi_{11} \not\hat{=} 0$ the equations (A12) and (A11) hold and they read,

$$\mathrm{Im}\Big[B(\bar{\delta} - \alpha + \bar{\beta})\Psi^1 - \frac{\kappa_{(\ell')}}{\kappa_{(\ell)}}(\bar{\delta} - \alpha + \bar{\beta})\Psi^0 - \kappa_{(\ell')}\Psi^1\overline{\Psi^0}\Big] \;\hat{=}\; 0 \tag{B70}$$

$$\begin{aligned} &\mathrm{Re}\Big[B\Big((\bar{\delta} + 2\pi - \alpha + \bar{\beta})\Psi^1 + \kappa_{(\ell)}|\Psi^1|^2 + \mu^1\Big) - \\ &\frac{\kappa_{(\ell')}}{\kappa_{(\ell)}}(\bar{\delta} + 2\pi - \alpha + \bar{\beta})\Psi^0 - \kappa_{(\ell)}\overline{\Psi^1}\Psi_0 - \frac{\kappa_{(\ell')}}{\kappa_{(\ell)}}\mu^0\Big] \;\hat{=}\; 0, \end{aligned} \tag{B71}$$

$$\begin{aligned} &B\Big((\bar{\delta} + 2\pi + \alpha - \bar{\beta})\overline{\Psi^1} + \kappa_{(\ell)}\overline{\Psi^1}^2 + \lambda^1\Big) \\ &+\kappa_{(\ell')}\overline{\Psi^1\Psi^0} - \frac{\kappa_{(\ell')}}{\kappa_{(\ell)}}(\bar{\delta} + 2\pi + \alpha - \bar{\beta})\overline{\Psi^0} - \frac{\kappa_{(\ell')}}{\kappa_{(\ell)}}\lambda^0 \;\hat{=}\; 0. \end{aligned} \tag{B72}$$

These are a set of necessary conditions on the horizon geometry for an IH structure to exist.

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